Characteristic polynomial: examples

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8.13: $\underbrace{f(z)}_{\text{entire}} = \underbrace{g(z)}_{\text{entire}} \cdot \underbrace{\overset{n}{p(z)} + \overset{n-1}{q(z)}}_{\text{polys}}$.

8.14: $\dfrac{d^j q}{dz^j}(\lambda_k) = \dfrac{d^j f}{dz^j}(\lambda_k)$.

Ex 8.15:

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

A has 2 evals: -1 and -3. We know $p_A(\lambda)$ has degree 2 (why?), so q is linear. We want $e^A = q(A)$, where

$$q(z) = az + b. \leftarrow$$

Here $f = e^z$. The conditions by 8.14 are

$$q(-1) = e^{-1}. \qquad q(-3) = e^{-3}.$$

$$-a + b = e^{-1}$$
$$-3a + b = e^{-3}$$

$$a = \tfrac{1}{2}(e^{-1} - e^{-3})$$

$$b = \tfrac{1}{2}(3e^{-1} - e^{-3}).$$

Now, we put everything together:

$$e^A = q(A)$$
$$= a \cdot A + b \cdot I$$

$e^{\cdots} = q(A)$

$= a\cdot A + b\cdot I$

$= \frac{1}{2}(e^{-1}-e^{-3})\begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix} + \frac{1}{2}(3e^{-1}-e^{-3})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$= \frac{1}{2}e^{-1}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \frac{1}{2}e^{-3}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}. \quad \square$

Ex 8.16: $A = \begin{pmatrix}1 & 1 & 4\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$. 3×3

The char. poly of A is $p_A(\lambda) = (\lambda - 1)^3$.
Thus, 1 eigenvalue with mult. 3. The poly. q is going to have degree 2.

$$q(z) = \underline{a}z^2 + \underline{b}z + \underline{c}.$$

$q(1) = e \qquad a + b + c = e$

$q'(1) = e \Rightarrow 2a + b = e$

$q''(1) = e \qquad 2a = e$

Here, $a = e/2$, $b = 0$, $c = e/2$.

$$q(z) = \frac{e}{2}(z^2 + 1).$$

$$e^A = q(A) = \frac{e}{2}(A^2 + I)$$

$$= \frac{e}{2}\left(\begin{pmatrix}1 & 1 & 4\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}^2 + \begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}\right)$$

$$= \frac{e}{2}\begin{pmatrix}2 & 2 & 9\\ 0 & 2 & 2\\ \cdots\end{pmatrix}$$

$\begin{pmatrix}1 & 1 & 4\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 4\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 9\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$

$$= \frac{e}{2}\begin{pmatrix} 2 & 2 & 9 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

$$= e\begin{pmatrix} 1 & 1 & 9/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Question: Determine e^A of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -3 & -1 \\ 1 & 4 & 2 \end{pmatrix}.$$