# Analysis of linear ODE systems

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**Prop 8.17:** If $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$, then the unique solution to the IVP

$$x' = Ax$$

$$x(t_0) = x_0$$

is given by

$$x(t) = \underline{e^{A(t-t_0)}x_0}.$$

**Proof:** By Prop 8.4,

$$x' = \frac{d}{dt}\left(e^{A(t-t_0)}x_0\right)$$

$$= A \cdot \underline{e^{A(t-t_0)}x_0}$$

$$= A \cdot x.$$

Initial condi

$$x(t_0) = e^{A(t_0-t_0)}x_0 = x_0.$$

Now we show uniqueness. For $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$ $(f(t,x) = Ax)$

$$\|f(t,x) - f(t,y)\| = \underline{\|Ax - Ay\|}$$

$$\underline{\leq \|A\| \cdot \|x - y\|}$$

Since $A$ is fixed, $\|A\|$ is some constant. Thus, $f$ is L.c. on $\mathbb{R}\times\mathbb{R}^d$. The sol. is unique by Thm 7.9. ☒

The solutions to the system in Prop 8.17 define a flow.

$$\Phi(t, x_0) = e^{At} x_0.$$

$\Phi(0, x_0) = e^{A\cdot 0} x_0 = x_0.$ ✓

$$\Phi(s+t, x_0) = e^{A(s+t)} x_0 = e^{As} e^{At} x_0$$
$$= e^{As} \Phi(t, x_0)$$
$$= \Phi(s, \Phi(t, x_0)).$$

The fixed points of the system are $x=0$ and any eigenvectors whose eigenvalue is 0.

**Ex 8.18:**

$$x_1' = x_1 + 2x_2$$
$$x_2' = 2x_1 + 4x_2$$

where $x(1) = (3, -1)^T$.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

All sol. have the form

$$x(t) = e^{A(t-1)} \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

A has 2 evals: 0 and 5 with evecs

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$S = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$

$\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$

$$S = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \checkmark$$

$$tA = S\underline{tD}S^{-1}$$

$$= \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 5t \end{pmatrix}\frac{1}{5}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

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$$e^{tA} = \frac{1}{5}\begin{pmatrix} e^{5t}+4 & 2e^{5t}-2 \\ 2e^{5t}-2 & 4e^{5t}+1 \end{pmatrix}$$

Thus, the solution is:

$$x(t) = \frac{1}{5}\begin{pmatrix} e^{5(t-1)}+4 & 2e^{5(t-1)}-2 \\ 2e^{5(t-1)}-2 & 4e^{5(t-1)}+1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} e^{5(t-1)}+14 \\ 2e^{5(t-1)}-7 \end{pmatrix}.$$

$$\Phi(t, x_0) = \frac{1}{5}\begin{pmatrix} e^{5t}+4 & 2e^{5t}-2 \\ 2e^{5t}-2 & 4e^{5t}+1 \end{pmatrix} x_0 .$$

Show $c\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is fixed.

$$\boxed{\Phi(t, cx_0) = cx_0 .}$$

$$\Phi\left(t, c\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right) = \frac{1}{5}\begin{pmatrix} e^{5t}+4 & 2e^{5t}-2 \\ 2e^{5t}-2 & 4e^{5t}+1 \end{pmatrix} c \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$= \frac{c}{5}\begin{pmatrix} -10 \\ 5 \end{pmatrix}$$

$$\left( -2 \right.$$

$$= c\begin{pmatrix}-2\\1\end{pmatrix}.$$

The entire subspace spanned by $\begin{pmatrix}-2\\1\end{pmatrix}$ is fixed. $\square$

**Thm 8.19** If $A \in \mathrm{Mat}_{d\times d}(\mathbb{R})$ and it has $d$ linearly independent eigenvectors $v_1, \dots, v_d$ with evals $\lambda_1, \dots, \lambda_d$, then $\exists$ constants $c_1, \dots, c_d$ s.t. the sol. to $x' = Ax$ is

$$x(t) = c_1 e^{\lambda_1 t} v_1 + \dots + c_d e^{\lambda_d t} v_d.$$

Proof: In lecture notes.

**Ex 8.20:** $A = \begin{pmatrix}1&2\\2&4\end{pmatrix}$, where $x(1) = \begin{pmatrix}3\\-1\end{pmatrix}$.

We have 2 L.I. evecs $\begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$, so by thm 8.19:

$$x(t) = \underline{\underline{c_1}}\, e^{0\cdot t}\begin{pmatrix}-2\\1\end{pmatrix} + \underline{\underline{c_2}}\, e^{5t}\begin{pmatrix}1\\2\end{pmatrix}.$$

Use the initial condition

$$x(1) = \begin{pmatrix}3\\-1\end{pmatrix}.$$

$$\begin{pmatrix}3\\-1\end{pmatrix} = x(1) = \underline{\underline{c_1}}\begin{pmatrix}-2\\1\end{pmatrix} + \underline{\underline{c_2}}\, e^{5}\begin{pmatrix}1\\2\end{pmatrix}$$

$$\boxed{\begin{aligned} 3 &= -2c_1 + e^5 c_2 \\ -1 &= c_1 + 2e^5 c_2 \end{aligned}}$$

**Thm 8.22:** The fixed pt $x=0$ of the system $x' = Ax$ is asymptotically stable if and only if $\mathrm{Re}(\lambda) < 0$ for all evals $\lambda$ of $A$.

*prof.* If $A$ is diagonalizable, then from thm 8.19

$$\rightarrow \quad x(t) = c_1 e^{\lambda_1 t} v_1 + \dots + c_d e^{\lambda_d t} v_d.$$

Note $|e^{\lambda t}| = e^{\mathrm{Re}(\lambda) t}$. Thus $e^{\mathrm{Re}(\lambda) t} \to 0$ if and only $\mathrm{Re}(\lambda) < 0$. If true for all $\lambda$, then $\|x\| \to 0$ as $t \to \infty$. If $A$ is not diagonalizable, then the poly will have terms of the form $t^i e^{\lambda t}$. However this does not change the abs. value. $\square$

**Ex 8.23:**

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

$$x(t) = \begin{pmatrix} e^{\lambda_1 t} x_0^{(1)} \\ e^{\lambda_2 t} x_0^{(2)} \end{pmatrix}.$$

Case 1: $\lambda_1 < 0, \lambda_2 < 0$

**Case 1: $\lambda_1 < 0, \lambda_2 < 0$**

By thm. $x = 0$ is stable & asymp. st.

**Case 2: $\lambda_1 > 0, \lambda_2 > 0$**

By thm 8.19, $x=0$ is unst. and repelling.

**Case 3: $\lambda_1 > 0, \lambda_2 < 0$**

The $x=0$ is a saddle ptn and unst.
The x-axis is unst/repelling - manifold
and the y-axis the the stable - manifold.

**Case 4: $\lambda_1 = 0, \lambda_2 > 0$**

The x-axis is fixed, while the y-axis
is an unst. manifold.

**Case 5: $\lambda_1 = 0, \lambda_2 < 0$**

The x-axis is fixed, while the y-axis
is a stable - manifold.

**Ex 8.24:** $A = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}$, $x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

**Ex 8.25:** $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ $x(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.
$\lambda \in \mathbb{R}$

**Ex 8.26:** Complex eigenvalues.

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \alpha I + \beta \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{J}.$$

$J$

$$e^{A} = e^{\alpha I + \beta J} = \underbrace{e^{\alpha I}}_{\text{easy}} \underbrace{e^{\beta J}}_{\text{harder}}.$$

$$e^{\beta J} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

$$e^{At} = e^{\alpha I t} e^{\beta J t} = e^{\alpha t}\begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix}.$$

$$\boxed{\alpha = 2, \quad \beta = 1}.$$

**Question:** What is the behavior of the system?