Multilinear tools for groups

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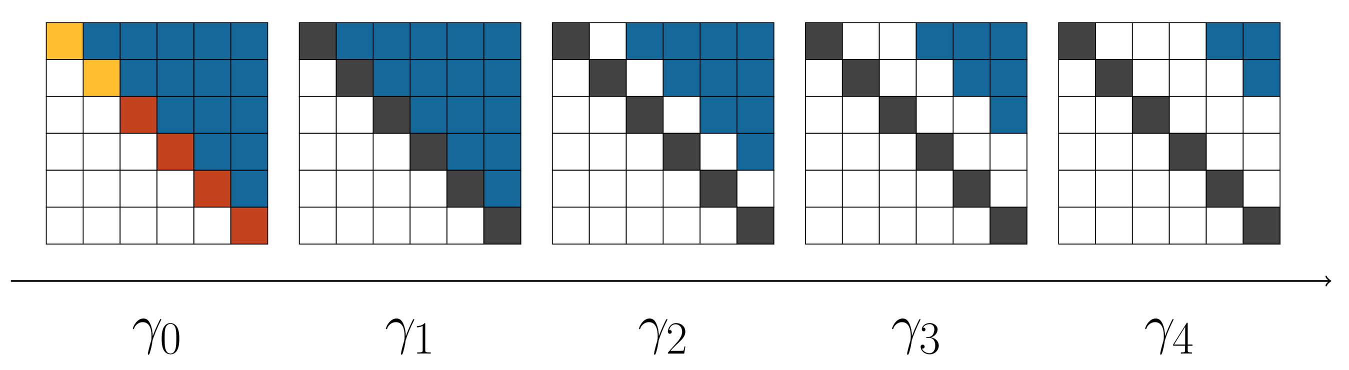
Structure in nilpotent groups

We cannot easily find characteristic subgroups of nilpotent groups, but we know they exist in abundance.

Consider the lower central series of upper unitriangular matrices,

$$\gamma_0 = G,\ \gamma_1 = \mathrm{Fit}(G), \qquad (\forall s \geq 1) \quad \gamma_{s+1} = [\gamma_s, \gamma_1].$$



Set $L_0(\gamma) = 0$ and for $s \neq 0$, $L_s(\gamma) = \gamma_s/\gamma_{s+1}$.

There is an $\mathbb{N}$-graded Lie K-algebra and $K[\gamma_0/\gamma_1]$-module:

$$L(\gamma) = \bigoplus L_s(\gamma) \cong K^5 \oplus K^4 \oplus K^3 \oplus K^2 \oplus K.$$

The grading of $L(\gamma)$ has K-bilinear maps from the commutator:

$$[,]_{s,t} : L_s(\gamma) \times L_t(\gamma) \rightarrowtail L_{s+t}(\gamma)$$

Algebras associated to bilinear maps

Suppose $\circ : U \times V \rightarrowtail W$ is a bilinear map of K-vector spaces.

With X, Y, and Z as linear operators, some algebras associated to $\circ$ are:

$$\mathcal{C}(\circ) = \{(X, Y, Z) \mid (uX) \circ v = u \circ (vY) = (u \circ v)Z\},$$

$$\mathcal{L}(\circ) = \{(X, Z) \mid (uX) \circ v = (u \circ v)Z\},$$

$$\mathcal{M}(\circ) = \left\{(X, Y^t) \mid (uX) \circ v = u \circ (Yv)\right\},$$

$$\mathcal{R}(\circ) = \{(Y, Z) \mid u \circ (vY) = (u \circ v)Z\},$$

$$\mathrm{Der}(\circ) = \{(X, Y, Z) \mid (uX) \circ v + u \circ (vY) = (u \circ v)Z\}.$$

All algebras above are efficiently computed using linear algebra.

If $\circ$ comes from a group, like $[,]_{s,t}$, then $\mathrm{Aut}(G)$ acts on all the above algebras.

The algebra $\mathrm{Der}(\circ)$ is a Lie algebra, while others are associative.

Multilinear algebra Magma software available on GitHub and

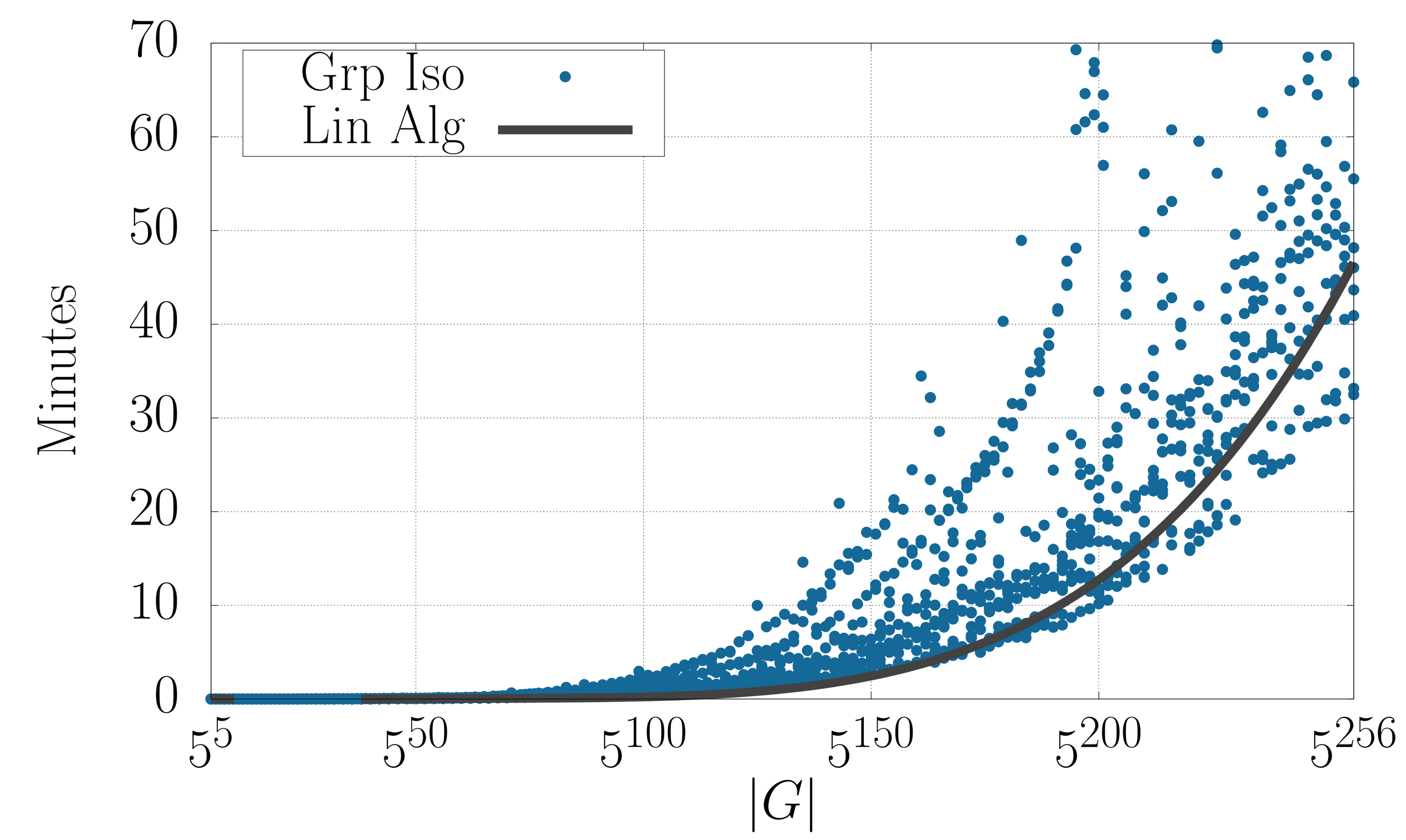
`https://thetensor.space`

Highly efficient isomorphism test

For some classes of p-groups, the algebra associated to $[,]_{1,1}$ contains so much structure that it nearly determines the isomorphism classes.

Theorem 1 ([BMW2]). *For finite, odd p-groups G_1 and G_2 of exponent p and class 2, there exists polynomial-time algorithms to decide*

1. *if there exists field K such that $L(\gamma(G_i)) \cong K^n \oplus K^2$ is a K-algebra, and if so*
2. *decide if $G_1 \cong G_2$.*



Geometry explains exact sequences

Associate ideals of $K[x, y, z]$ from definitions of algebras, see [FMW]:

$$\mathscr{I}(\circ, \mathcal{C}(\circ)) = (x - y, y - z), \qquad \mathscr{I}(\circ, \mathcal{R}(\circ)) = (y - z),$$

$$\mathscr{I}(\circ, \mathcal{L}(\circ)) = (x - z), \qquad \mathscr{I}(\circ, \mathrm{Der}(\circ)) = (x + y - z).$$

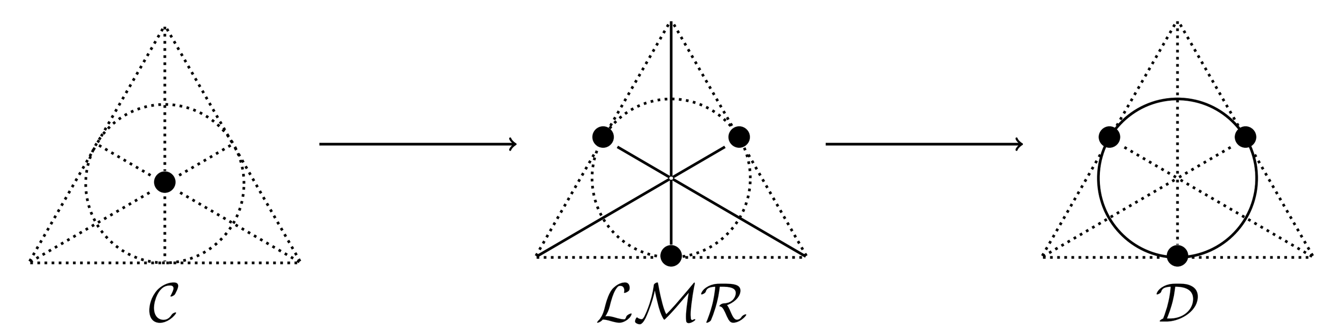
$$\mathscr{I}(\circ, \mathcal{M}(\circ)) = (x - y),$$

Theorem 2 ([BMW1]). *Suppose $\circ : U \times V \to W$ is non-degenerate. For $\mathcal{LMR} = \mathcal{L}(\circ) \oplus \mathcal{M}(\circ) \oplus \mathcal{R}(\circ)$, the following are exact sequences of Lie algebras and groups, respectively:*

$$0 \longrightarrow \mathcal{C}(\circ) \longrightarrow \mathcal{LMR}(\circ) \longrightarrow \mathrm{Der}(\circ),$$

$$0 \longrightarrow \mathcal{C}(\circ)^\times \longrightarrow \mathcal{LMR}(\circ)^\times \longrightarrow \mathrm{Aut}(\circ).$$

Sequences in Theorem 2 hold for multilinear maps, based on geometry.



Use filters to refine and recurse

An $(\mathbb{N}^d, G)$-*filter* is a function from the pre-ordered monoid $\mathbb{N}^d$ into the set of normal subgroups of G such that for all $s, t \in \mathbb{N}^d$,

$$[\phi_s, \phi_t] \leq \phi_{s+t}, \qquad s \preceq t \implies \phi_s \geq \phi_t.$$

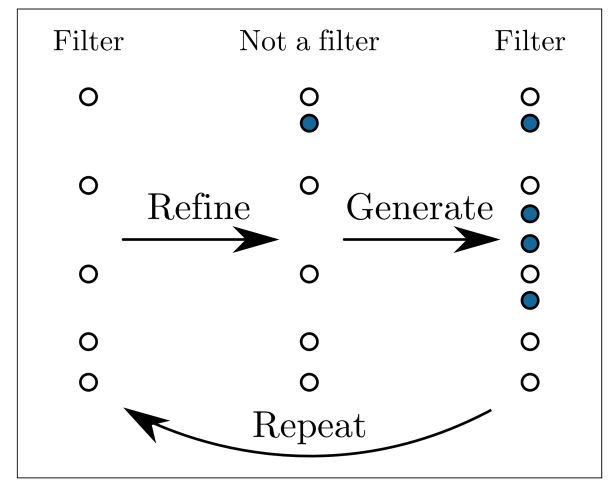
The lower central series, γ, for G is an $(\mathbb{N}, G)$-filter.

Theorem 3 ([M2]). *If ϕ is a totally ordered $(\mathbb{N}^d, G)$-filter and $H \lhd G$ refines ϕ, then there exists an efficient algorithm (polynomial time in $\log |G|$) that constructs a filter from ϕ including H.*

Refining filters change:

- the monoid and grading,
- algebras associated to bilinear maps,
- the isomorphism problem.

Recursively find new structure and simplify isomorphism, see [M1].

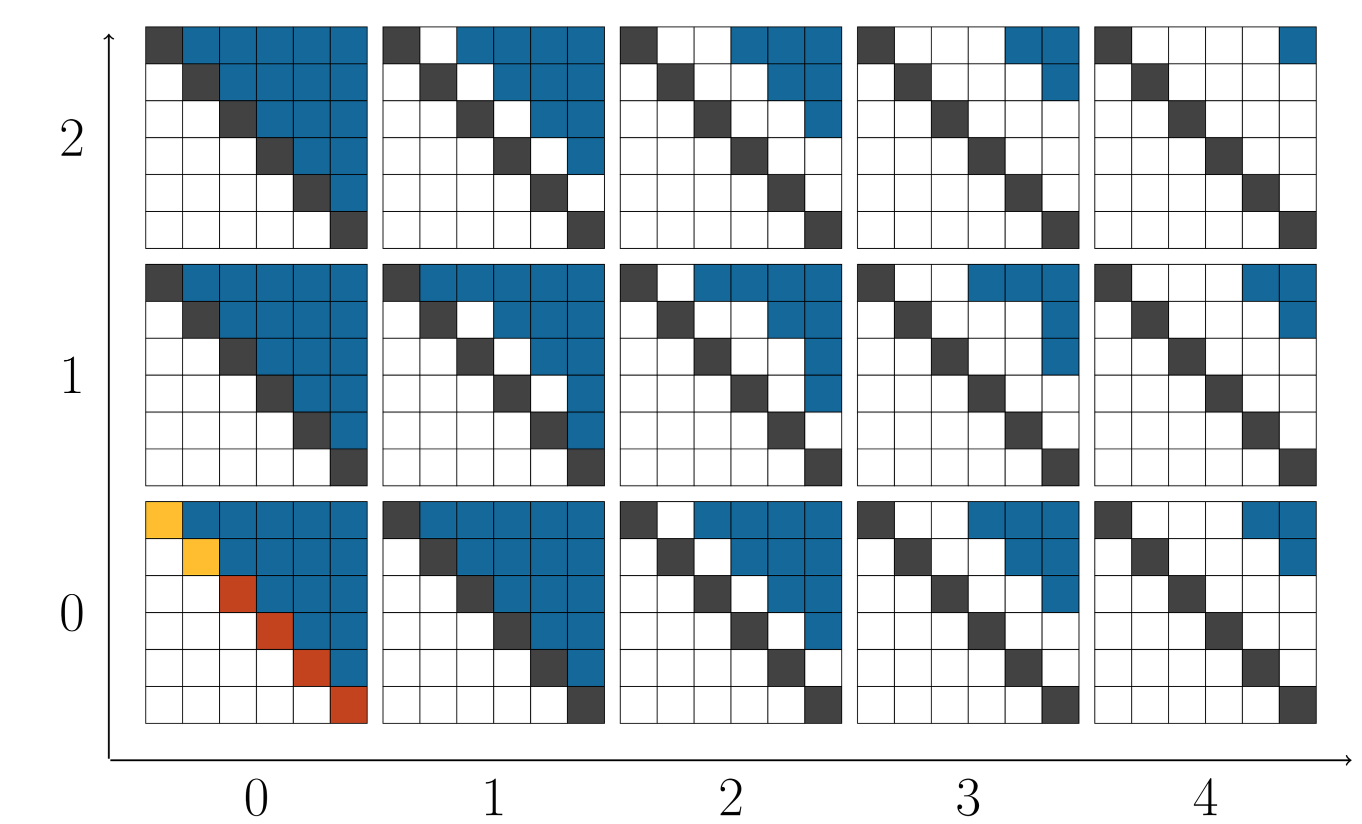


An abundance of refinements

Refine filters from features in algebras associated to grading of $L(\gamma)$.

From one refinement, we define an $(\mathbb{N}^2, G)$-filter ϕ shown below, see [M3].

$$L(\phi) = (K^3 \oplus K^2) \oplus (K^2 \oplus K^2) \oplus (K^2 \oplus K) \oplus K^2 \oplus K$$



References

[BMW1] P. A. Brooksbank, J. Maglione, and J. B. Wilson, *Exact sequences of inner automorphisms of tensors*, J. Algebra **545** (2020), 43–63.

[BMW2] ______, *A fast isomorphism test for groups whose Lie algebra has genus 2*, J. Algebra **473** (2017), 545–590.

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[M2] ______, *Efficient characteristic refinements for finite groups*, J. Symbolic Comput. **80** (2017), no. 2, 511–520.

[M3] ______, *Longer nilpotent series for classical unipotent subgroups*, J. Group Theory **18** (2015), no. 4, 569–585.