Lattices and tableaux: Exercises

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Problems with lower numbers are generally easier. Problems marked as (Challenging) require significant effort or background knowledge. Those for which I do not know the solution or proof are marked as (Open).

1. (a) Let $T = (C_1, C_2, \dots) \in \mathrm{SSYT}_n$ and $T' = ([n], C_1, C_2, \dots)$. Prove that

   $$\Phi_{T'}(Y) = \Phi_T(Y).$$

   (b) Let $T \in \mathrm{SSYT}_n$, and let $\mathrm{r}(T)$ be the reduced tableau obtained from T by removing all repeated columns. Prove that

   $$\Phi_{\mathrm{r}(T)}(Y) = \Phi_T(Y).$$

2. We define the following tableaux:

$$T_1 = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 3 & 4 \\ \hline 2 & 4 & 5 & 5 \\ \hline 4 & 6 & 6 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array} \qquad T_2 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 4 & 4 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 4 & 4 \\ \cline{1-2} \end{array} \qquad T_3 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 5 & 7 & 7 & 8 \\ \hline 3 & 3 & 6 & 8 \\ \cline{1-4} 4 & 4 & 7 \\ \cline{1-3} 8 \\ \cline{1-1} \end{array}$$

   (a) Compute $\Phi_{T_i}(Y)$ for each i.

   (b) Determine the flag of partitions $\lambda^{\bullet}(T_i)$ for each i.

3. Consider the following matrix

$$\begin{pmatrix} p^3 & 0 & 0 & p \\ & 1 & 1 & 0 \\ & & p & p-1 \\ & & & p^2 \end{pmatrix} \in \mathrm{Mat}_4(\mathbb{Z}_p).$$

   (a) What is the associated intersection tableau?

   (b) What is the associated projection tableau?

4. Suppose $M \in \mathrm{Mat}_n(\mathbb{Z}_p)$ has associated intersection tableau $T \in \mathrm{SSYT}_n$. What is the intersection tableau associated with pM?

5. Prove that

$$\mathsf{HLS}_2(Y, X_1, X_2, X_{12}) = \frac{1 - YX_1X_2}{(1-X_1)(1-X_2)(1-X_{12})}$$

6. Let λ and μ be partitions of n. Let G be a finite abelian p-group of type λ and $H \leqslant G$ of type μ. If G/H is cyclic, then prove that $\mu \subseteq \lambda$ and $\lambda - \mu$ is a horizontal strip.

   [Hint: consider the subgroup $\Omega_k(G) = \{g \in G \mid p^k g = 0\}$ for $k \in \mathbb{N}$.]

7. A tableau $T \in \mathrm{rSSYT}_n$ is **maximal** if T has exactly $\binom{n+1}{2}$ columns. List four of the twelve maximal tableaux of rSSYT_4. Which subsets of $[n]$ *must* be columns of a maximal tableau of rSSYT_n?

8. Prove that

$$\mathrm{affS}^{\mathrm{in}}_{2,\mathbb{Z}_p}(Z_{11}, Z_{21}, Z_{22}) = \frac{1 - Z_{11}Z_{21}}{(1-pZ_{11})(1-Z_{21})(1-Z_{11}Z_{22})},$$

$$\mathrm{affS}^{\mathrm{pr}}_{2,\mathbb{Z}_p}(Z_{11}, Z_{21}, Z_{22}) = \frac{1 - Z_{11}Z_{21}}{(1-Z_{11})(1-pZ_{21})(1-Z_{11}Z_{22})}.$$

9. Characterize the class of tableaux $T \in \mathrm{SSYT}_n$ satisfying $\Phi_T(Y) = 1$.

10. Determine the set of lattices in $\mathbb{Z}_p^3$ with the following intersection tableau.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

    What are the cardinalities?

11. Do Problem 10 but replace "intersection" with "projection".

12. Show that two matrices with the same intersection tableau need not have the same projection tableau.

    [Hint: consider the sets obtained from Problem 10.]

13. Recall that the following is a complete isolated flag:

$$0 = V^{(0)} < V^{(1)} < \cdots < V^{(n-1)} < V^{(n)} = \mathbb{Z}_p^n,$$

    and for a lattice $\Lambda \leqslant \mathbb{Z}_p^n$, recall that $\lambda^{(i)}(\Lambda)$ is the type of $V^{(i)}/(\Lambda \cap V^{(i)})$ for all $i \in [n]$. Prove that $\lambda^{(i)}(\Lambda) \subseteq \lambda^{(i+1)}(\Lambda)$ and $\lambda^{(i+1)}(\Lambda) - \lambda^{(i)}(\Lambda)$ is a horizontal strip for all $i \in [n-1]$.

14. (Challenging) Let $T \in \mathrm{rSSYT}_n$ and $N = \binom{n+1}{2}$. Prove that T is maximal (see Problem 7 for definition) if and only if

$$\left\{ \lambda_j^{(i)}(T) \;\middle|\; i \in [n], j \in [i] \right\} = [N].$$

15. (Challenging) Prove Birkhoff's formula: for partitions $\mu \subseteq \lambda$, the number of subgroups of $C_\lambda(\mathbb{Z}_p)$ isomorphic to $C_\mu(\mathbb{Z}_p)$ is

$$\prod_{i\in\mathbb{N}} p^{\mu'_{i+1}(\lambda'_i-\mu'_i)} \binom{\lambda'_i - \mu'_{i+1}}{\mu'_i - \mu'_{i+1}}_p = \prod_{i\in\mathbb{N}} p^{\mu'_i(\lambda'_i-\mu'_i)} \binom{\lambda'_i - \mu'_{i+1}}{\lambda'_i - \mu'_i}_{p^{-1}}$$

where λ' is the conjugate partition obtained by transposing the associated Young diagram. (Comment: I think the left side is easier to prove, but we used the right side.)

[Hint: Consider the subgroups Ω_k from the hint in Problem 6.]

16. (Challenging) Let $s_\lambda(\boldsymbol{x})$ be the Schur polynomial on $\boldsymbol{x} = (x_1, \dots, x_n)$ associated with the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathcal{P}_n$. For $C \subseteq [n]$, let $\boldsymbol{x}_C = \prod_{i\in C} x_i$. Prove the following identity:

$$\mathsf{HLS}_n(0, (\boldsymbol{x}_C)_C) = \sum_{\lambda\in\mathcal{P}_n} s_\lambda(\boldsymbol{x}).$$

17. (a) (Open) Prove the following conjecture. For all $n \in \mathbb{N}$,

$$\max_{T\in\mathrm{SSYT}_n} \deg \Phi_T(Y) = \binom{n}{2},$$

and the unique polynomial attaining this maximal degree is $(1-Y)^{\binom{n}{2}}$.

(b) (Open) Assume $\deg \Phi_T(Y) = \binom{n}{2}$, how many unique columns can T have? For $n = 3$, it is either 5 or 6.

18. (Open) Characterize the class of tableaux $T \in \mathrm{SSYT}_n$ satisfying $\Phi_T(-1) = 0$. (This is Problem 8.6 of Maglione–Voll (arXiv:2410.08075).)