Refinement and general filters for groups

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What is intrinsic to a group?

Main question: What structure is intrinsic to a group G?

A group G given with the shapes:

$$\begin{bmatrix} 1 & A \\ 0 & I_{12} \end{bmatrix}, \qquad \begin{bmatrix} I_2 & B \\ 0 & I_6 \end{bmatrix}, \qquad \begin{bmatrix} I_3 & C \\ 0 & I_4 \end{bmatrix}.$$

A group given with the shapes:

$$\begin{bmatrix} 1 & A & C \\ & I_2 & B \\ & & I_8 \end{bmatrix} \qquad \begin{bmatrix} I_2 & X & Z \\ & I_3 & Y \\ & & I_4 \end{bmatrix}$$

What about scalars?

$$G = \begin{bmatrix} 1 & A & C \\ & I_2 & B \\ & & I_8 \end{bmatrix} \qquad H = \begin{bmatrix} I_2 & X & Z \\ & I_3 & Y \\ & & I_4 \end{bmatrix}$$

Verbal subgroups produce a series:

$$\begin{bmatrix} 1 & A & C \\ & I_2 & B \\ & & I_8 \end{bmatrix} > \begin{bmatrix} 1 & & C \\ & I_2 & \\ & & I_8 \end{bmatrix} > \begin{bmatrix} 1 & & \\ & I_2 & \\ & & I_8 \end{bmatrix},$$

$$\begin{bmatrix} I_2 & X & Z \\ & I_3 & Y \\ & & I_4 \end{bmatrix} > \begin{bmatrix} I_2 & & Z \\ & I_3 & \\ & & I_4 \end{bmatrix} > \begin{bmatrix} I_2 & & \\ & I_3 & \\ & & I_4 \end{bmatrix}.$$

In both groups, $\gamma_1/\gamma_2 \cong K^{16}, \quad \gamma_2 \cong K^8$.

In some cases, the shape **is** intrinsic

For a field K, let $$H_{abc}(K) = \left\{ \begin{bmatrix} I_a & X & Z \\ & I_b & Y \\ & & I_c \end{bmatrix} \;\middle|\; \begin{array}{c} X \in M_{ab}(K) \\ Y \in M_{bc}(K) \\ Z \in M_{ac}(K) \end{array} \right\}.$$

Theorem (J.B. Wilson 2017)

For groups $H_{abc}(K)$, the integers a, b, c are isomorphism invariants, and they can be computed in polynomial time.

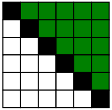
Now: special case of a larger body of work with U. First, J.B. Wilson.

Idea: All that information found in algebras associated to

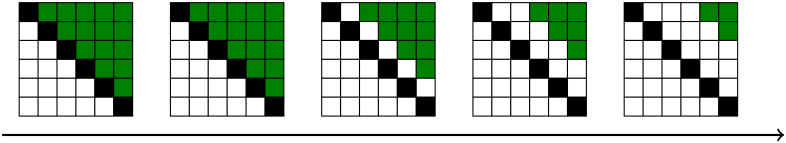
$$[,] : K^{ab+bc} \times K^{ab+bc} \rightarrowtail K^{ac}.$$

Larger examples are refinable

$$G = \begin{bmatrix} 1 & * & * & * & * & * \\ & 1 & * & * & * & * \\ & & 1 & * & * & * \\ & & & 1 & * & * \\ & & & & 1 & * \\ & & & & & 1 \end{bmatrix} =$$



E.g. $\gamma_0 = \gamma_1 = G$ and $\gamma_{s+1} = [\gamma_s, \gamma_1]$, for $s \geq 1$.



γ_0 γ_1 γ_2 γ_3 γ_4

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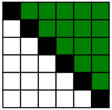

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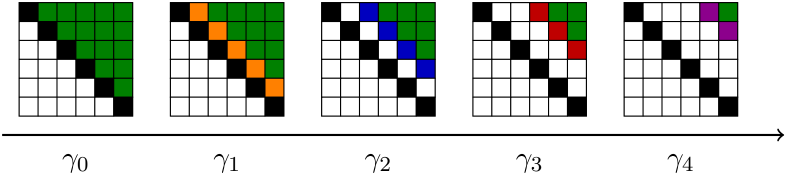


$$L(\gamma) = K^5 \oplus K^4 \oplus K^3 \oplus K^2 \oplus K$$

Filters produce refinable graded Lie algebras

A *filter* is a function $\phi : \langle \mathbb{N}^d, \preceq \rangle \to 2^G$ into the normal subgroups with

$$[\phi_s, \phi_t] \leq \phi_{s+t} \qquad \text{and} \qquad s \preceq t \text{ implies } \phi_s \geq \phi_t.$$

Theorem (J.B. Wilson 2013)

If $\phi : \mathbb{N}^d \to 2^G$ is a filter, then

$$L(\phi) = \bigoplus_{s \neq 0} \phi_s \,/ \langle \phi_{s+t} \mid t \neq 0 \rangle$$

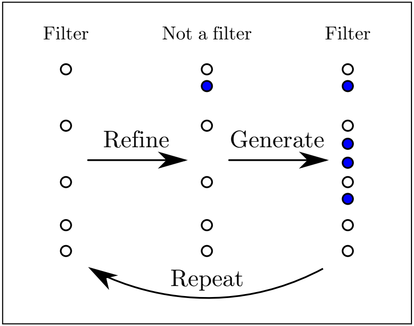
is an $\mathbb{N}^d$-graded Lie ring. Each graded ideal lifts to a filter refinement.

Efficient refinements for filters

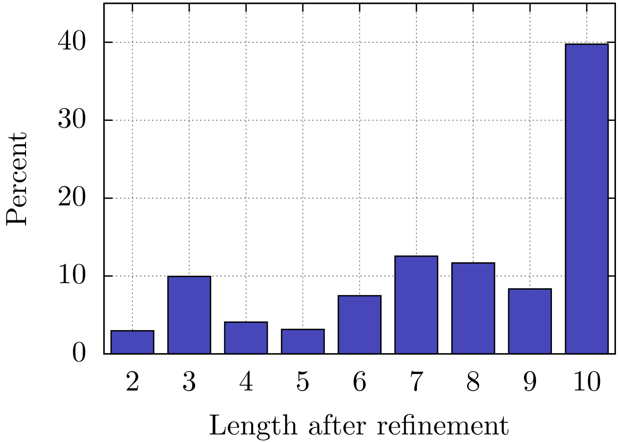
Theorem (M. 2017)

If $\phi : \mathbb{N}^d \to 2^G$ is a totally ordered filter and $H \triangleleft G$ refines ϕ, then there exists an efficient algorithm (polynomial time in $\log |G|$) that constructs a filter from ϕ including H.

- Provides structure that connects $\mathbb{N}^d$ to subgroups of G that can be updated.
- Allows for efficient recursion.



Survey of 500,000,000 groups of order 2^{10}



Filters uncover new characteristic structure [**M.-Wilson**].

Refining the algebra to get smaller steps

$$\phi^{(1)} : \mathbb{N} \to 2^G \qquad L\left(\phi^{(1)}\right) = \bigoplus_{s\neq 0} L_s$$

Refine

$$\phi^{(2)} : \mathbb{N}^2 \to 2^G \qquad L\left(\phi^{(2)}\right) = \bigoplus_{s\neq 0} L_s$$

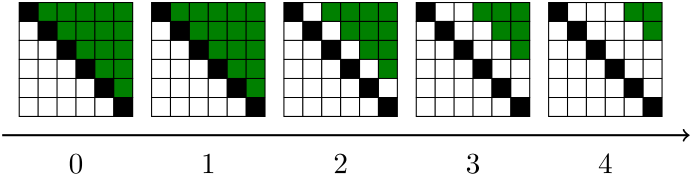
Refine

$$\phi^{(3)} : \mathbb{N}^3 \to 2^G \qquad L\left(\phi^{(3)}\right) = \bigoplus_{s\neq 0} L_s$$

Refine

$$\phi : \mathbb{N}^d \to 2^G \qquad L(\phi) = \bigoplus_{s\neq 0} L_s$$

Refinement improves even well-known examples



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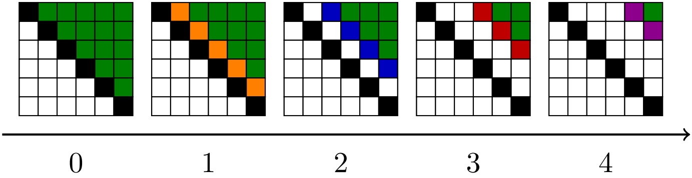
$$L(\gamma) = K^5 \oplus K^4 \oplus K^3 \oplus K^2 \oplus K$$



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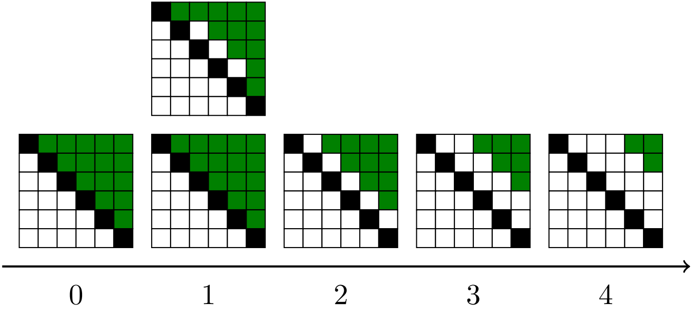

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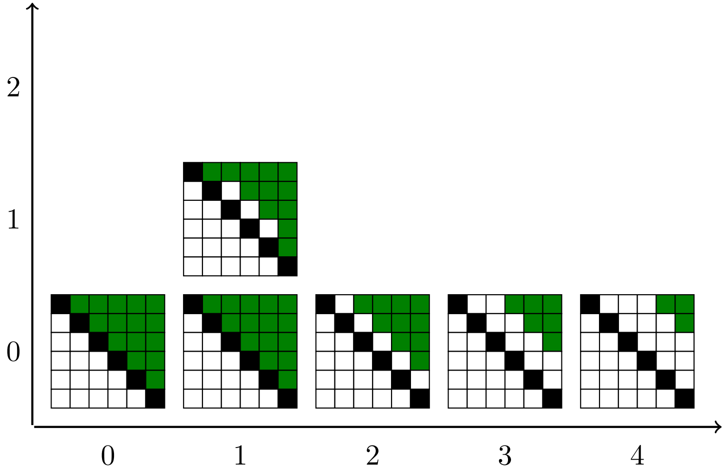

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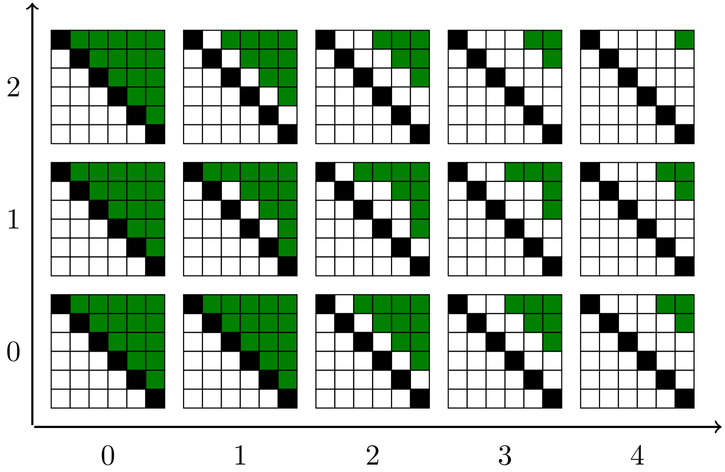

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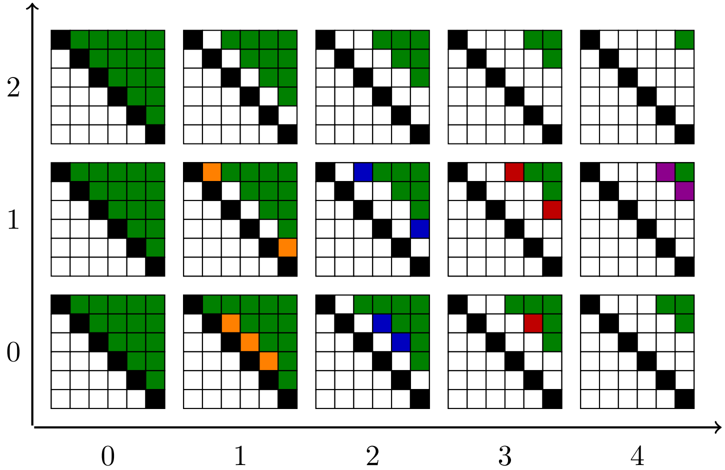
$L(\gamma) = K^5 \oplus K^4 \oplus K^3 \oplus K^2 \oplus K$

$L(\phi) = K^3 \oplus K^2 \oplus K^2 \oplus K^2 \oplus K \oplus K^2 \oplus K^2 \oplus K$



Use module and ring theory to start refinement

- Suppose $\circ : U \times V \rightarrowtail W$ is a bilinear map of K-vector spaces.

- Some algebras associated to $\circ$ are

$$\begin{aligned}
\mathcal{L}_\circ &= \{(X, Z) \mid (Xu) \circ v = Z(u \circ v)\}, \\
\mathcal{M}_\circ &= \{(X, Y) \mid (uX) \circ v = u \circ (Yv)\}, \\
\mathcal{R}_\circ &= \{(Y, Z) \mid u \circ (vY) = (u \circ v)Z\}, \\
\mathrm{Cent}(\circ) &= \{(X, Y, Z) \mid (uX) \circ v = u \circ (vY) = (u \circ v)Z\}, \\
\mathrm{Der}(\circ) &= \{(X, Y, Z) \mid (uX) \circ v + u \circ (vY) = (u \circ v)Z\}.
\end{aligned}$$

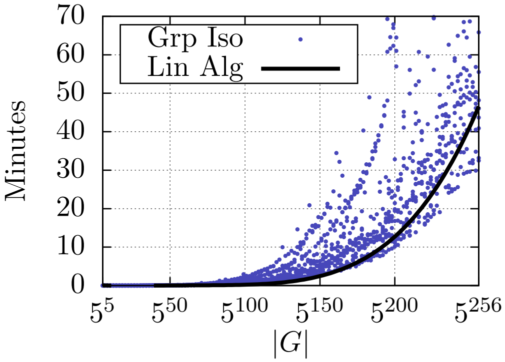
- Ongoing work with Brooksbank and Wilson using representation theory of Lie algebras in the context of isomorphism problems.

- Multilinear Algebra package for Magma on GitHub [**M.-Wilson**].

Looking for structure in new places

Theorem (Brooksbank-M.-Wilson, 2017)

There exists a polynomial-time algorithm to test isomorphism of groups of exponent p with central commutator subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.



- Used by Brooksbank, O'Brien, and Wilson to efficiently search for local structure.
- Implemented in MAGMA.

Study the group through $L(\phi)$

Two fundamental problems arise in partially-ordered case:

- Let G be nilpotent, and $\gamma : \mathbb{N} \to 2^G$ the lower central series.

  Set $\phi : \mathbb{N}^2 \to 2^G$ such that for $s = (s_1, s_2) \in \mathbb{N}^2$,

  $$\phi_s = \gamma_{s_1}.$$

  The associated Lie algebra is trivial

  $$L(\phi) = \bigoplus_{s \neq 0} \phi_s / \langle \phi_{s+t} \mid t \neq 0 \rangle \quad = 0.$$

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Theorem (M. 2018)

If G is nilpotent and $\phi : \mathbb{N}^d \to 2^G$ is a filter, then there exists a filter $\theta : \mathbb{N}^d \to 2^G$ such that

- *$im(\phi) \subseteq im(\theta)$ and*
- *there is a surjection $L(\theta) \to G$.*

The other problem

- Let K be a field of order q and $G = \left\{ \begin{bmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{bmatrix} : a, b, c \in K \right\}$.

  There are $q+1$ distinct subgroups $G' < H < G$.

  There is a filter $\phi : \mathbb{N}^{q+1} \to 2^G$, such that

$$\dim L(\phi) = q + 2.$$

A bijection between G and $L(\phi)$ is recovered

A filter $\phi : \mathbb{N}^d \to 2^G$ is *compatible* if there exists $\mathcal{X} \subset G$:

1. $G = \langle \mathcal{X} \rangle$,
2. for all $s \in \mathbb{N}^d$, $\langle \phi_s \cap \mathcal{X} \rangle = \phi_s$,
3. $H \mapsto H \cap \mathcal{X}$ is a complete lattice embedding from $\mathrm{im}(\phi)$ to $2^{\mathcal{X}}$,
4. for all $x \in \mathcal{X}$, there exists a unique $s \in \mathbb{N}^d$ such that $x \in \phi_s \backslash \langle \phi_{s+t} \mid t \neq 0 \rangle$.

Theorem (M. 2018)

Suppose G is nilpotent and polycyclic. If $\phi : \mathbb{N}^d \to 2^G$ is compatible, then there exists a bijection between the set of bases for $L(\phi)$ and the polycyclic generating sets for G.

Filters provide different location for structure

- Filters refine many examples of groups: 97% in survey of 2^{10}.
- Algebras associated to bilinear maps $L_s \times L_t \rightarrowtail L_{s+t}$.
- Structure from entire $\mathbb{N}^d$-graded $L(\phi)$.
- Developed for isomorphism, but are general tools.