

Multilinear tools through filters on groups

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What is intrinsic to a group?

Main question: What structure is intrinsic to a group G ?

A group G given with the shapes:

$$\begin{bmatrix} 1 & A \\ 0 & I_{12} \end{bmatrix}, \quad \begin{bmatrix} I_2 & B \\ 0 & I_6 \end{bmatrix}, \quad \begin{bmatrix} I_3 & C \\ 0 & I_4 \end{bmatrix}.$$

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A group given with the shapes:

$$\begin{bmatrix} 1 & A & C \\ & I_2 & B \\ & & I_8 \end{bmatrix} \quad \begin{bmatrix} I_2 & X & Z \\ & I_3 & Y \\ & & I_4 \end{bmatrix}$$

What about scalars?

$$G = \begin{bmatrix} 1 & A & C \\ & I_2 & B \\ & & I_8 \end{bmatrix} \quad H = \begin{bmatrix} I_2 & X & Z \\ & I_3 & Y \\ & & I_4 \end{bmatrix}$$

Verbal subgroups produce a series:

$$\begin{bmatrix} 1 & A & C \\ & I_2 & B \\ & & I_8 \end{bmatrix} > \begin{bmatrix} 1 & & C \\ & I_2 & \\ & & I_8 \end{bmatrix} > \begin{bmatrix} 1 & & \\ & I_2 & \\ & & I_8 \end{bmatrix},$$

$$\begin{bmatrix} I_2 & X & Z \\ & I_3 & Y \\ & & I_4 \end{bmatrix} > \begin{bmatrix} I_2 & & Z \\ & I_3 & \\ & & I_4 \end{bmatrix} > \begin{bmatrix} I_2 & & \\ & I_3 & \\ & & I_4 \end{bmatrix}.$$

In both groups, $\gamma_1/\gamma_2 \cong K^{16}$, $\gamma_2 \cong K^8$.

In some cases, the shape is intrinsic

$$\text{For a field } K, \text{ let } H_{abc}(K) = \left\{ \left[\begin{array}{ccc} I_a & X & Z \\ & I_b & Y \\ & & I_c \end{array} \right] \mid \begin{array}{l} X \in M_{ab}(K) \\ Y \in M_{bc}(K) \\ Z \in M_{ac}(K) \end{array} \right\}.$$

Theorem (J.B. Wilson 2017)

For groups $H_{abc}(K)$, the integers a, b, c are isomorphism invariants, and they can be computed in polynomial time.

Now: special case of a larger body of work with U. First, J.B. Wilson.

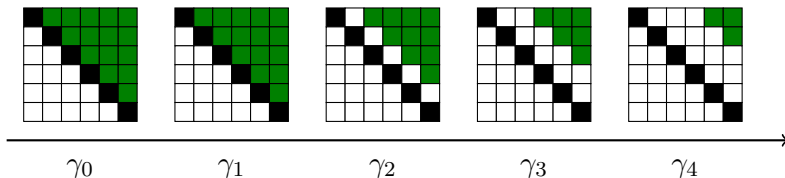
Idea: All that information found in algebras associated to

$$\begin{aligned} [\cdot, \cdot] &: \gamma_1/\gamma_2 \times \gamma_1/\gamma_2 \twoheadrightarrow \gamma_2, \\ [\cdot, \cdot] &: K^{ab+bc} \times K^{ab+bc} \twoheadrightarrow K^{ac}. \end{aligned}$$

Larger examples are refinable

$$G = \begin{bmatrix} 1 & * & * & * & * & * \\ & 1 & * & * & * & * \\ & & 1 & * & * & * \\ & & & 1 & * & * \\ & & & & 1 & * \\ & & & & & 1 \end{bmatrix} = \begin{array}{|c|c|c|c|c|c|} \hline \blacksquare & \color{green}\blacksquare & \color{green}\blacksquare & \color{green}\blacksquare & \color{green}\blacksquare & \color{green}\blacksquare \\ \hline \blacksquare & \blacksquare & \color{green}\blacksquare & \color{green}\blacksquare & \color{green}\blacksquare & \color{green}\blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \color{green}\blacksquare & \color{green}\blacksquare & \color{green}\blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \color{green}\blacksquare & \color{green}\blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \color{green}\blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}$$

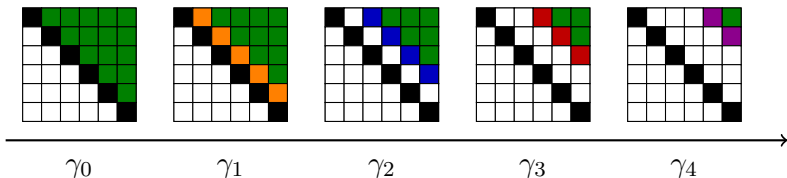
E.g. $\gamma_0 = \gamma_1 = G$ and $\gamma_{s+1} = [\gamma_s, \gamma_1]$, for $s \geq 1$.



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E.g. $\gamma_0 = \gamma_1 = G$ and $\gamma_{s+1} = [\gamma_s, \gamma_1]$, for $s \geq 1$.



$$L(\gamma) = K^5 \oplus K^4 \oplus K^3 \oplus K^2 \oplus K$$

Filters produce refinable graded Lie algebras

A *filter* is a function $\phi : \langle \mathbb{N}^d, \preceq \rangle \rightarrow 2^G$ into the normal subgroups with

$$[\phi_s, \phi_t] \leq \phi_{s+t} \quad \text{and} \quad s \preceq t \text{ implies } \phi_s \geq \phi_t.$$

Theorem (J.B. Wilson 2013)

If $\phi : \mathbb{N}^d \rightarrow 2^G$ is a filter, then

$$L(\phi) = \bigoplus_{s \neq 0} \phi_s / \langle \phi_{s+t} \mid t \neq 0 \rangle$$

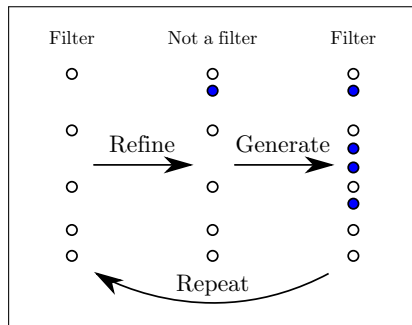
is an \mathbb{N}^d -graded Lie ring. Each graded ideal lifts to a filter refinement.

Efficient refinements for filters

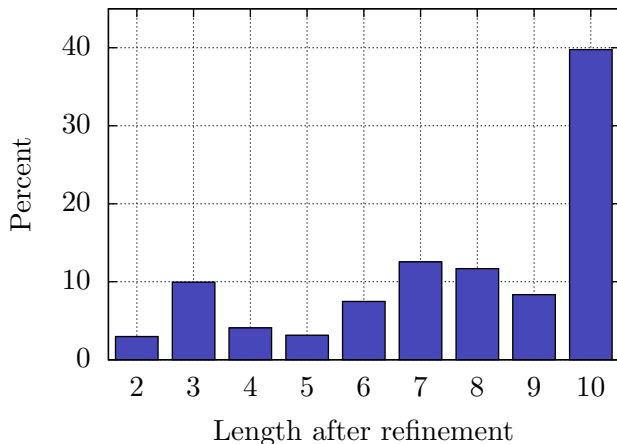
Theorem (M. 2017)

If $\phi : \mathbb{N}^d \rightarrow 2^G$ is a totally ordered filter and $H \triangleleft G$ refines ϕ , then there exists an efficient algorithm (polynomial time in $\log |G|$) that constructs a filter from ϕ including H .

- Provides structure that connects \mathbb{N}^d to subgroups of G that can be updated.
- Allows for efficient recursion.

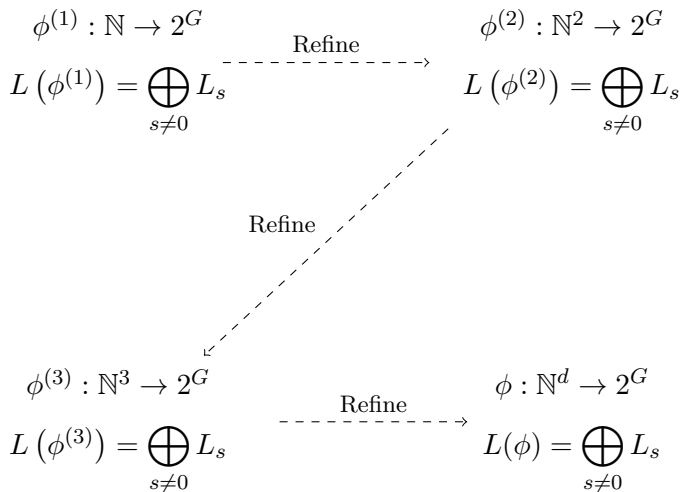


Survey of 500,000,000 groups of order 2^{10}

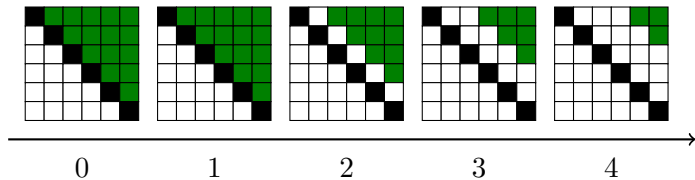


Filters uncover new characteristic structure [M.-Wilson].

Refining the algebra to get smaller steps

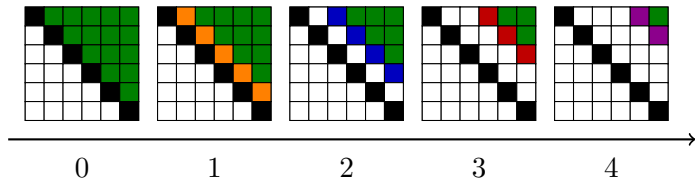


Refinement improves even well-known examples



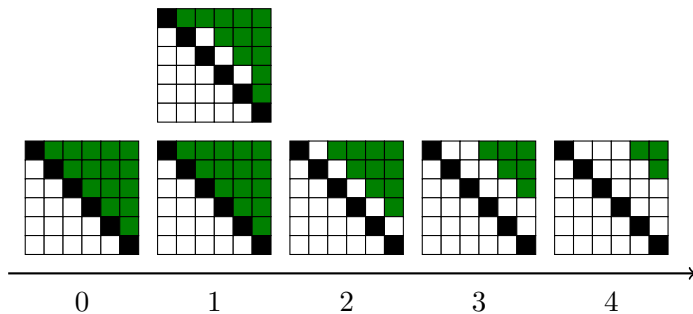
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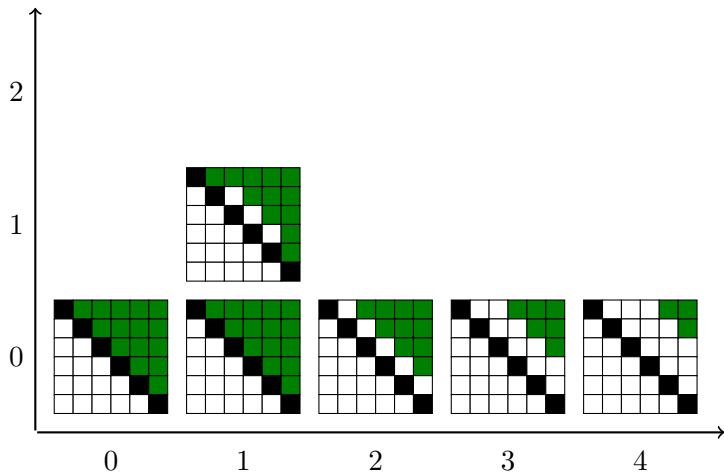
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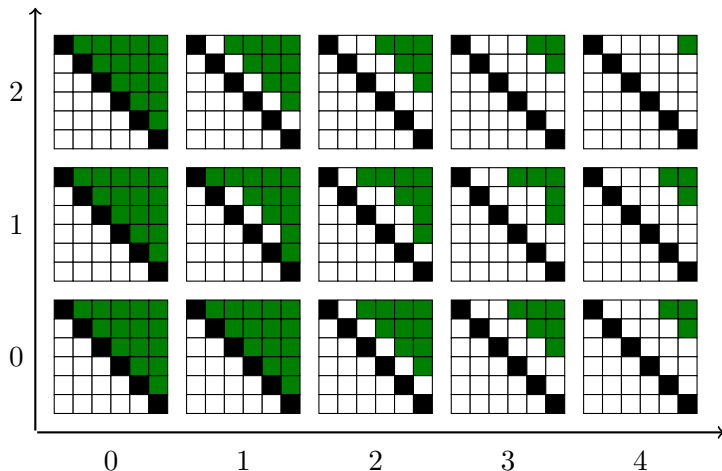
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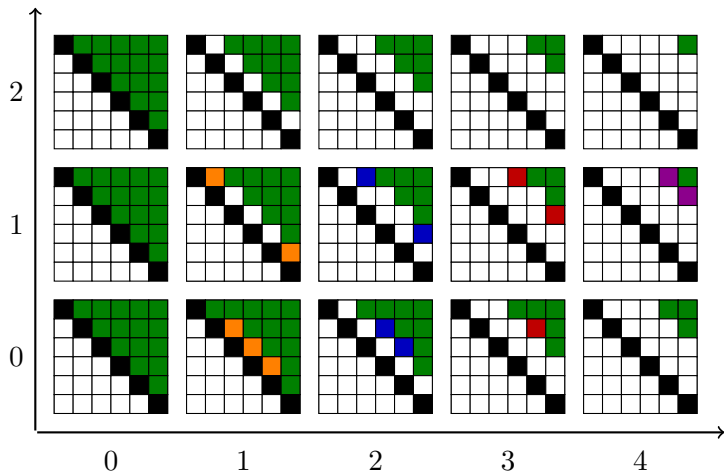
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Refinement improves even well-known examples

$$L(\gamma) = K^5 \oplus K^4 \oplus K^3 \oplus K^2 \oplus K$$

$$L(\phi) = K^3 \oplus K^2 \oplus K^2 \oplus K^2 \oplus K \oplus K^2 \oplus K^2 \oplus K$$



Use module and ring theory to start refinement

- Suppose $\circ : U \times V \rightarrow W$ is a bilinear map of K -vector spaces.
- Some algebras associated to \circ are

$$\mathcal{L}_\circ = \{(X, Z) \mid (Xu) \circ v = Z(u \circ v)\},$$

$$\mathcal{M}_\circ = \{(X, Y) \mid (uX) \circ v = u \circ (Yv)\},$$

$$\mathcal{R}_\circ = \{(Y, Z) \mid u \circ (vY) = (u \circ v)Z\},$$

$$\text{Cent}(\circ) = \{(X, Y, Z) \mid (uX) \circ v = u \circ (vY) = (u \circ v)Z\},$$

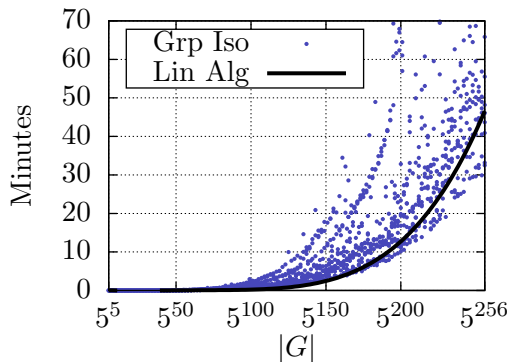
$$\text{Der}(\circ) = \{(X, Y, Z) \mid (uX) \circ v + u \circ (vY) = (u \circ v)Z\}.$$

- Ongoing work with Brooksbank and Wilson using representation theory of Lie algebras in the context of isomorphism problems.
- Multilinear Algebra package for MAGMA on GitHub [**M.-Wilson**].

Looking for structure in new places

Theorem (Brooksbank-M.-Wilson, 2017)

There exists a polynomial-time algorithm to test isomorphism of groups of exponent p with central commutator subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.



- Used by Brooksbank, O'Brien, and Wilson to efficiently search for local structure.
- Implemented in MAGMA.

Study the group through $L(\phi)$

Two fundamental problems arise in partially-ordered case:

- Let G be nilpotent, and $\gamma : \mathbb{N} \rightarrow 2^G$ the lower central series.

Set $\phi : \mathbb{N}^2 \rightarrow 2^G$ such that for $s = (s_1, s_2) \in \mathbb{N}^2$,

$$\phi_s = \gamma_{s_1}.$$

The associated Lie algebra is trivial

$$L(\phi) = \bigoplus_{s \neq 0} \phi_s / \langle \phi_{s+t} \mid t \neq 0 \rangle = 0.$$

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Theorem (M. 2018)

If G is nilpotent and $\phi : \mathbb{N}^d \rightarrow 2^G$ is a filter, then there exists a filter $\theta : \mathbb{N}^d \rightarrow 2^G$ such that

- *$\text{im}(\phi) \subseteq \text{im}(\theta)$ and*
- *there is a surjection $L(\theta) \rightarrow G$.*

The other problem

- Let K be a field of order q and $G = \left\{ \begin{bmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{bmatrix} : a, b, c \in K \right\}$.

There are $q + 1$ distinct subgroups $G' < H < G$.

There is a filter $\phi : \mathbb{N}^{q+1} \rightarrow 2^G$, such that

$$\dim L(\phi) = q + 2.$$

A bijection between G and $L(\phi)$ is recovered

A filter $\phi : \mathbb{N}^d \rightarrow 2^G$ is *compatible* if there exists $\mathcal{X} \subset G$:

- 1 $G = \langle \mathcal{X} \rangle$,
- 2 for all $s \in \mathbb{N}^d$, $\langle \phi_s \cap \mathcal{X} \rangle = \phi_s$,
- 3 $H \mapsto H \cap \mathcal{X}$ is a complete lattice embedding from $\text{im}(\phi)$ to $2^{\mathcal{X}}$,
- 4 for all $x \in \mathcal{X}$, there exists a unique $s \in \mathbb{N}^d$ such that $x \in \phi_s \setminus \langle \phi_{s+t} \mid t \neq 0 \rangle$.

Theorem (M. 2018)

Suppose G is nilpotent and polycyclic. If $\phi : \mathbb{N}^d \rightarrow 2^G$ is compatible, then there exists a bijection between the set of bases for $L(\phi)$ and the polycyclic generating sets for G .

Filters provide different location for structure

- Filters refine many examples of groups: 97% in survey of 2^{10} .
- Algebras associated to bilinear maps $L_s \times L_t \mapsto L_{s+t}$.
- Structure from entire \mathbb{N}^d -graded $L(\phi)$.
- Developed for isomorphism, but are general tools.