EXACT SEQUENCES OF INNER AUTOMORPHISMS OF TENSORS

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Abstract. We produce a long exact sequence whose terms are unit groups of associative algebras that behave as inner automorphisms of a given tensor. Our sequence generalizes known sequences for associative and non-associative algebras. In a manner similar to those, our sequence facilitates inductive reasoning about, and calculation of the groups of symmetries of a tensor. The new insights these methods afford can be applied to problems ranging from understanding algebraic structures to distinguishing entangled states in particle physics.

In memory of C.C. Sims.

1. Introduction

The purpose of this work is to provide tools to expose the symmetries of a tensor. By a *tensor* we mean a vector, t, that can be interpreted as a multilinear map $\langle t|\colon U_{\vav} \times \cdots \times U_1 \rightarrowtail U_0$. For instance, a $(d_2 \times d_1 \times d_0)$-grid, $t = [t_{ij}^k]$, of scalars may be interpreted as a multilinear map $\langle t| : \mathbb{K}^{d_2} \times \mathbb{K}^{d_1} \rightarrowtail \mathbb{K}^{d_0}$ evaluated on $|u_2, u_1\rangle \in \mathbb{K}^{d_2} \times \mathbb{K}^{d_1}$ as follows:

$$\langle t|u_2, u_1\rangle = \left(\sum_{i=1}^{d_2} \sum_{j=1}^{d_1} u_{2i} t_{ij}^1 u_{1j}, \dots, \sum_{i=1}^{d_2} \sum_{j=1}^{d_1} u_{2i} t_{ij}^{d_0} u_{1j} \right) \in \mathbb{K}^{d_0}.$$

Throughout, $\rightarrowtail$ will denote a multilinear map, while $\to$ will be reserved for linear maps. Tensors describe diverse structures, including distributive products in algebra, affine connections in differential geometry, quantum entanglement in particle physics, and measurements and meta-data in statistical models.

A natural objective in the study of a tensor $\langle t|\colon U_{\vav} \times \cdots \times U_1 \rightarrowtail U_0$ is to discover properties invariant under change of basis. We are particularly interested in one such invariant, namely its group

(1.1)
$$\mathrm{Aut}(t) = \left\{ \alpha \in \prod_{a=0}^{\vav} \mathrm{Aut}_{\mathbb{K}}(U_a) \;\middle|\; \forall u \in \prod_a U_a,\ \alpha_0 \langle t|u_{\vav}, \dots, u_1\rangle = \langle t|\alpha_{\vav} u_{\vav}, \dots, \alpha_1 u_1\rangle \right\}$$

of symmetries. This group is difficult to compute so it is helpful to break it up into an exact sequence of groups which are, in general, easier to construct.

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There are precedents in algebra for such an approach. The main idea, for a given associative algebra A, is to study $\mathrm{Aut}(A)$ by placing it in an exact sequence

$$1 \to \mathrm{Inn}(A) \to \mathrm{Aut}(A) \to \mathcal{J}(A), \tag{1.2}$$

where $\mathrm{Inn}(A)$ is the group of inner automorphisms, and $\mathcal{J}(A)$ has a natural representation that can be explored without knowing $\mathrm{Aut}(A)$. For instance, in Skolem-Noether type theorems, $\mathcal{J}(A)$ is the group of Galois automorphisms of the center of A. General *Rosenberg–Zelinsky sequences* relate $\mathcal{J}(A)$ to the module theory of A via ideal class groups, Picard groups, and so forth [AH, GM, BFRS].

Variations for non-associative Lie and Jordan algebras were carried out by Jacobson and others [BO]. Here, the notion of "inner" automorphisms is not obvious. If L is a Lie algebra, for example, the appropriate substitutes are from the units in the associative algebra generated by left actions $L_x \in \mathrm{End}(L)$, where $L_x(y) = [x, y]$. Factoring out the inner automorphisms of central simple algebras leaves a group naturally represented as a Galois group—and in more general cases, a group akin to the $\mathcal{J}(A)$ used in general Rosenberg-Zelinsky sequences.

For tensors, the concept of inner automorphisms is even less clear. Even for a bilinear map $\langle t| \colon U_2 \times U_1 \rightarrowtail U_0$, the left actions $L_x \in \mathrm{Hom}(U_1, U_0)$ compose only when $U_0 = U_1$. For tensors of higher valence, there are $\binom{\vav+1}{2}$ suitable analogues of "left" or "right" actions. A solution however is visible in an earlier extension of Skolem-Noether type theorems for bilinear maps by the third author [W]; cf. Section 6.2. The result is that our sequences do not begin $0 \to \mathrm{Inn}(t) \to \mathrm{Aut}(t)$ but rather extend to the left of $\mathrm{Aut}(t)$ with a long sequence of corrections. Once in place, one can essentially follow the treatment in [W] for non-associative algebras to extend the sequences to the right.

1.1. **Notation & terminology.** Throughout, the Hebrew letter $\vav$ (*vav* to evoke *valence*) is a nonnegative integer. Set $[\vav] = \{0, \dots, \vav\}$ and $\binom{[\vav]}{i} = \{A \subset [\vav] \mid |A| = i\}$. For $A \subset [\vav]$, write $\bar{A} = [\vav] - A$ and $\bar{a} = [\vav] - \{a\}$. Let $\mathbb{K}$ be a commutative unital ring and let $U_\vav, \dots, U_0$ be finitely generated $\mathbb{K}$-modules. Define

$$U_0 \oslash U_1 = \mathrm{Hom}(U_1, U_0) \qquad U_0 \oslash \cdots \oslash U_\vav = (U_0 \oslash \cdots \oslash U_{\vav-1}) \oslash U_\vav.$$

Then $(-) \oslash U_a$ is a functor on modules with (left) adjoint functor $(-) \otimes U_a$ giving rise to the following natural isomorphisms of $\mathbb{K}$-modules:

$$\begin{aligned} U_0 \oslash (U_\vav \otimes \cdots \otimes U_1) &\cong U_0 \oslash \cdots \oslash U_{a-1} \oslash (U_\vav \otimes \cdots \otimes U_a) \\ &\cong U_0 \oslash \cdots \oslash U_a \oslash (U_\vav \otimes \cdots \otimes U_{a+1}) \\ &\cong U_0 \oslash \cdots \oslash U_\vav. \end{aligned}$$

A *tensor space* is a $\mathbb{K}$-module T equipped with a $\mathbb{K}$-module monomorphism

$$\langle \cdot | : T \hookrightarrow U_0 \oslash \cdots \oslash U_\vav. \tag{1.3}$$

An element $t \in T$ is a *tensor*, and $\langle t| \colon U_\vav \times \cdots \times U_1 \rightarrowtail U_0$ is its associated multilinear map. For $|u\rangle = |u_\vav, \dots, u_1\rangle \in \prod_{a \neq 0} U_a$, write $\langle t|u\rangle \in U_0$ to mean the evaluation of $\langle t|$ at $|u\rangle$. The set $\{U_0, \dots, U_\vav\}$ of modules is the *frame* of T, and $\vav$ is its *valence*. For brevity, we often write $S \subseteq U_0 \oslash \cdots \oslash U_\vav$ to denote a set of tensors and its frame.

Put $\Omega := \prod_{a \in [\vav]} \mathrm{End}(U_a)$, the ring of *transverse operators* on the tensor space $U_0 \oslash \cdots \oslash U_\vav$, where $\mathrm{End}(U_a)$ is the ring of $\mathbb{K}$-linear endomorphisms of U_a. The group $\prod_{a \in [\vav]} \mathrm{Aut}(U_a)$ of invertible transverse operators is the group $\Omega^\times$ of units of

Ω. For $\omega_a \in \mathrm{End}(U_a)$, write $|\omega_a u_a, u_{\bar{a}}\rangle$ to apply ω_a to u_a while leaving the other coordinates fixed. If, for each $a \in [\vav]$, the condition $\langle t | u_a, U_{\bar{a}}\rangle = 0$ implies $u_a = 0$, then t is *nondegenerate*; if $U_0 = \langle t | U_{\bar{0}}\rangle$ then t is *full*. We say t is *fully nondegenerate* if it is full and nondegenerate, and we lose no essential information by assuming all our tensors are of this type.

1.2. **Main results.** We adopt Albert's *autotopisms* and Leger–Luks' *generalized derivations* [LL] as the principal invariants to study. For $S \subset U_0 \oslash \cdots \oslash U_{\vav}$,

$$\mathrm{Der}(S) = \left\{ \omega \in \Omega \;\middle|\; \forall t \in S, \forall u \in \prod_{a\neq 0} U_a,\ \omega_0 \langle t \mid u\rangle = \sum_{c \in [\vav]-0} \langle t \mid \omega_c u_c, u_{\bar{c}}\rangle \right\} \tag{1.4}$$

is the (Lie) algebra of *derivations* of S, and

$$\mathrm{Aut}(S) = \left\{ \alpha \in \Omega^{\times} \;\middle|\; \forall t \in S, \forall u \in \prod_{a\neq 0} U_a,\ \alpha_0 \langle t \mid u\rangle = \langle t \mid \alpha_{\bar{0}} u_{\bar{0}}\rangle \right\} \tag{1.5}$$

is the group of *automorphisms* (also called *autotopisms*) of S. For $0 < a < b \leqslant \vav$, put $\Omega_{0a} = \mathrm{End}(U_0) \times \mathrm{End}(U_a)$, $\Omega_{ab} = \mathrm{End}(U_a)^{\mathrm{op}} \times \mathrm{End}(U_b)$, and define

$$\begin{aligned} \mathrm{Nuc}_{ab}(S) &= \left\{ \omega \in \Omega_{ab} \;\middle|\; \forall t \in S,\ \forall u \in \prod_{c\neq 0} U_c,\ \langle t \mid \omega_a u_a, u_{\bar{a}}\rangle = \langle t \mid \omega_b u_b, u_{\bar{b}}\rangle \right\}, \\ \mathrm{Nuc}_{0a}(S) &= \left\{ \omega \in \Omega_{0a} \;\middle|\; \forall t \in S,\ \forall u \in \prod_{c\neq 0} U_c,\ \omega_0 \langle t \mid u\rangle = \langle t \mid \omega_a u_a, u_{\bar{a}}\rangle \right\} \end{aligned} \tag{1.6}$$

the *nuclei* of S. The opposite ring in the first equation ensures that both types of nuclei are associative rings. For $A \subset [\vav]$, put $\Omega_A = \prod_{a\in A} \mathrm{End}(U_a)$. Define the *centroids* of S to be the associative rings

$$\begin{aligned} \mathrm{Cen}_A(S) &= \left\{ \omega \in \Omega_A \;\middle|\; \forall t \in S,\ \forall u \in \prod_{c\neq 0} U_c,\ \forall a, b \in A, \langle t \mid \omega_a u_a, u_{\bar{a}}\rangle = \langle t \mid \omega_b u_b, u_{\bar{b}}\rangle \right\}, \\ \mathrm{Cen}_{A\cup 0}(S) &= \left\{ \omega \in \Omega_{A\cup 0} \;\middle|\; \forall t \in S,\ \forall u \in \prod_{c\neq 0} U_c,\ \forall a \in A, \omega_0 \langle t \mid u\rangle = \langle t \mid \omega_a u_a, u_{\bar{a}}\rangle \right\}. \end{aligned} \tag{1.7}$$

where $A \subseteq [\vav] - 0$. The assumption that S is fully nondegenerate ensures that all centroids are commutative. For $2 < k \leqslant \vav$, put

$$\mathrm{Nuc}(S) = \bigoplus_{A \in \binom{[\vav]}{2}} \mathrm{Nuc}_A(S) \qquad \mathrm{Cen}_k(S) = \bigoplus_{A \in \binom{[\vav]}{k}} \mathrm{Cen}_A(S). \tag{1.8}$$

Our main theorems, which generalize Rosenberg–Zelinsky sequences to derivations and autotopisms of tensors, are the following.

Theorem A. *For each fully nondegenerate $S \subseteq U_0 \oslash \cdots \oslash U_{\vav}$, there is an exact sequence of $\mathbb{K}$-Lie algebras*

$$0 \to \mathrm{Cen}_{\vav}(S) \to \cdots \to \mathrm{Cen}_3(S) \to \mathrm{Nuc}(S) \to \mathrm{Der}(S).$$

Theorem B. *For each fully nondegenerate $S \subseteq U_0 \oslash \cdots \oslash U_{\vav}$, there is an exact sequence of groups*

$$1 \to \mathrm{Cen}_{\vav}(S)^{\times} \to \cdots \to \mathrm{Cen}_3(S)^{\times} \to \mathrm{Nuc}(S)^{\times} \to \mathrm{Aut}(S).$$

1.3. **Outline.** The paper is organized as follows. In Section 2, we describe more general operators than those used in the sequences of Theorems A and B. In particular we demonstrate how our theorems constitute part of a general strategy to attach polynomial-based invariants to tensors, which are then analyzed by restricting to subsets of variables. Geometrically, this corresponds to taking sections and inspecting fibers. Section 3 constructs the exact sequences in Theorems A and B, and identifies a certain combinatorial property needed to prove exactness. This property is examined in isolation in Section 4, followed by the proofs our main theorems in Section 5. Section 6 provides several examples that help to illustrate the breadth and power of our methods. These examples include detailed examinations of symmetries of tensors as well as a brief look at quantum particle entanglement, where two states are distinguished using our exact sequences.

2. Creating the sequences

Theorems A and B concern groups and Lie algebras, yet feature exact sequences whose terms are predominately associative algebras or their unit groups. Although this is a convenient conversion from a pragmatic viewpoint—associative algebras are better understood, structurally, than groups and Lie algebras—switching categories may seem to the reader unnatural. However, the conversion has a natural explanation when viewed through a broader geometric lens.

To introduce our sequences and expose their purpose we make use of a device introduced in [FMW] that records operators on a tensor space using polynomial identities. This establishes a convenient algebro-geometric vocabulary.

2.1. **Operator sets.** In [FMW] the following definitions where introduced as a means to capture, in a uniform manner, a wide range of common tensor operators. We include enough detail here to prove our stated claims in this more general context.

Let $\{U_0, \dots, U_{\vav}\}$ be a frame of $\mathbb{K}$-modules, $\Omega = \prod_{a\in[\vav]} \mathrm{End}(U_a)$ the ring of transverse operators on this frame, and $\mathbb{K}[X] = \mathbb{K}[x_{\vav}, \dots, x_0]$. For $\omega \in \Omega$, $p(X) = \sum_e \lambda_e X^e \in \mathbb{K}[X]$, $t \in U_{\vav} \oslash \cdots \oslash U_0$, and $|u\rangle = |u_{\vav}, \dots, u_1\rangle \in \prod_{a>0} U_a$, define

$$\langle t|p(\omega)|u\rangle = \sum_e \lambda_e \omega_0^{e_0} \langle t|\omega_{\vav}^{e_{\vav}} u_{\vav}, \dots, \omega_1^{e_1} u_1\rangle.$$

For each set S of tensors, and each $P \subset \mathbb{K}[X]$, let $\langle S|P(\omega)|U\rangle$ be the subspace generated by $\langle t|p(\omega)|u\rangle$ as t ranges over S, p over P, and $|u\rangle$ over $\prod_{a>0} U_a$. Then,

$$\mathfrak{Z}(S, P) = \left\{ \omega \in \prod_{a\in[\vav]} \mathrm{End}(U_a) \;\middle|\; \langle S|P(\omega)|U\rangle = 0 \right\} \tag{2.1}$$

is the *operator set* for the pair S, P. It will help our intuition to consider the sets $\mathfrak{Z}(S, P)$ as geometries. Indeed, over an algebraically closed field these are algebraic zero-sets. For other rings (such as finite fields, as required of several problems in algebra) $\mathfrak{Z}(S, P)$ still has the structure of an affine $\mathbb{K}$-scheme [FMW].

Remark 2.2. The polynomials defining $\mathfrak{Z}(S, P)$ as an affine $\mathbb{K}$-scheme are derived from the formula for P, but they are in general quite different from P. For instance, the number of variables in P is $\vav + 1$, whereas in general the polynomials describing $\mathfrak{Z}(S, P)$ involve $\sum_a (\dim U_a)^2$ variables. Thus, when defining a function on $\mathfrak{Z}(S, P)$, one should not expect its image or fibers to again be sets of the form $\mathfrak{Z}(S, P)$.

2.2. **Fibers of restriction.** We describe a general approach to study the zero-sets $\mathfrak{Z}(S, P)$ for an arbitrary set of polynomials; eventually, we return to the cases with which we are most concerned here. We begin by restricting the operator sets $\mathfrak{Z}(S, P)$ to subsets of the frame. Fix $A \subset [\vav]$, and define the projection

$$\Lambda_A \colon \mathfrak{Z}(S, P) \to \prod_{a\in A} \mathrm{End}(U_a), \qquad\qquad \Lambda_A(\omega_b \colon b \in [\vav]) = (\omega_a \colon a \in A).$$

Write $\mathfrak{Z}(S, P)|_A$ for the image of Λ_A, which as noted in Remark 2.2 need not be an operator set. However, the fibers of Λ_A are still comprised of operators that satisfy the polynomials P on tensors S. These we might describe as sets $\mathfrak{Z}(S, Q)$ for various Q related to P. Extending our notation slightly, for $\omega \in \mathfrak{Z}(S, P)$, write

$$\mathfrak{Z}(S, P(\omega_A, X_{\bar{A}})) = \Lambda_A^{-1}(\omega) = \{(\tau_{\bar{A}}, \omega_A) \mid \langle S|P(\tau_{\bar{A}}, \omega_A)|U\rangle = 0\}.$$

There creates two issues. First, the formula depends on ω_A. We can largely ignore this issues, however, since fibers over generic points—those not lying on a proper subvariety—are invariant for a fixed irreducible component. Thus, we have a notion of generic fibers over the components of $\mathfrak{Z}(S, P)$ that is independent of ω. Secondly, $P(\omega_A, X_{\bar{A}})$ is partially evaluated at linear operators and thus is no longer a polynomial in $\mathbb{K}[X]$. To get to a polynomial, it suffices to evaluate P at $\omega_A = (\lambda_a 1_{U_a} \mid a \in A)$ with $\lambda_a \in \mathbb{K}$. If ω_A is generic, its fibers are the operator sets

$$P_A(X_{\bar{A}}) = P(\lambda_A, X_{\bar{A}}) \in \mathbb{K}[X_{\bar{A}}].$$

In this way, a generic fiber is isomorphic to $\mathfrak{Z}(S, P_A)$ and can be regarded as an operator set, as in (2.1). Indeed, we will confine ourselves to cases where $\mathfrak{Z}(S, P)$ is a group and Λ_A is a group homomorphism; here, $\lambda_a \in \{0, 1\}$ will be a convenient choice for us to construct our sequences explicitly.

2.3. **Polynomials defining groups and algebras.** The following fact—which follows directly from definitions (1.4), (1.5) and (2.1)—concerns two specific polynomials related to the derivation algebra and the automorphism group of S.

Fact 2.3. *If* $D(X) = x_\vav + \cdots + x_1 - x_0$ *and* $G(X) = x_\vav \cdots x_1 - x_0$, *then*

$$\mathrm{Der}(S) = \mathfrak{Z}(S, D) \qquad\qquad \mathrm{Aut}(S) = \mathfrak{Z}(S, G).$$

Hereafter, we assume P is chosen so that $\mathfrak{Z}(S, P)$ is closed under one of two group operations: addition of endomorphisms, or composition of automorphisms. (A characterization of such P is given in [FMW], but it follows easily from Fact 2.3 that both $D(X)$ and $G(X)$ have this property.) In the latter case we still write $\mathfrak{Z}(S, P)$ but consider only invertible endomorphisms. Let ϵ denote the appropriate identity (0 or 1) for $\mathfrak{Z}(S, P)$, which we interpret naturally as a constant in $\mathbb{K}$. Each projection map, Λ_A is a group homomorphism. Thus, the fibration we created from Λ_A has a generic fiber, since every fiber is a coset of the kernel. In particular, for each $A \subseteq [\vav]$ we have an exact sequence

$$\{\epsilon\} \longrightarrow \mathfrak{Z}(S, P_A)|_A \longrightarrow \mathfrak{Z}(S, P) \xrightarrow{\Lambda_A} \mathfrak{Z}(S, P)|_A \longrightarrow \{\epsilon\}. \tag{2.4}$$

Translated into the language of Section 1, we observe the origins of our replacements for inner derivations and inner automorphisms.

Fact 2.5. *For each* $\varnothing \neq A \subseteq [\vav]$, *there are exact sequences*

$$\begin{array}{c} 0 \longrightarrow \operatorname{Der}_A(S) \hookrightarrow \operatorname{Der}(S) \xrightarrow{\Delta_A} \operatorname{Der}(S)|_A \longrightarrow 0 \\ , \\ 0 \longrightarrow \operatorname{Aut}_A(S) \hookrightarrow \operatorname{Aut}(S) \xrightarrow{\Gamma_A} \operatorname{Aut}(S)|_A \longrightarrow 0 \end{array}$$

where $\operatorname{Der}_A(S) = \mathfrak{Z}(S, D_A)|_A$ *and* $\operatorname{Aut}_A(S) = \mathfrak{Z}(S, G_A)|_A$. *(For emphasis, when specializing to derivations and autotopisms, shall replace the restriction maps* Λ *with* Δ *and* Γ*, respectively.)*

2.4. **Chains of derivations and automorphisms.** We obtain a global outlook by summing over all restrictions to sets of a common cardinality. In this way we have one parameter to consider instead of exponentially many subsets.

Fact 2.6. *For each* $k \in [\daleth]$*, there exists group homomorphisms* Λ_i^k $(i = 1, 2)$ *that make the following diagram commute, and ensure that* $\ker(\Lambda_1^k) = \operatorname{im}(\Lambda_2^k)$.

$$\begin{array}{ccccc} \coprod_{A \in \binom{[\daleth]}{k}} \mathfrak{Z}(S, P_A)|_A & \xrightarrow{\Lambda_2^k} & \mathfrak{Z}(S, P) & \xrightarrow{\Lambda_1^k} & \prod_{A \in \binom{[\daleth]}{k}} \mathfrak{Z}(S, P)|_A \\ \uparrow{\scriptstyle \iota_A} & \nearrow & & \searrow{\scriptstyle \Lambda_A} & \downarrow{\scriptstyle \pi_A} \\ \mathfrak{Z}(S, P_A)|_A & \longrightarrow & \{\epsilon\} & \longrightarrow & \mathfrak{Z}(S, P)|_A \end{array}$$

Fact 2.6 follows from the exact sequences in (2.4). Using $P = D$ we have the following corollary.

Fact 2.7. *There is an exact sequence*

$$\bigoplus_{A \in \binom{[\daleth]}{k}} \operatorname{Der}_A(S) \xrightarrow{\Delta_2^k} \operatorname{Der}(S) \xrightarrow{\Delta_1^k} \prod_{A \in \binom{[\daleth]}{k}} \operatorname{Der}(S)|_A$$

In formulating a group analogue we face the problem that finite coproducts of of groups are not isomorphic to products. This will prevent us from extending our sequence to the left in the case of $P = G$. The solution to the problem is implied by Theorem B, where we swapped from groups to units in a ring.

Lemma 2.8. *For* $0 \leqslant a < b \leqslant \daleth$ *there is a natural monomorphism*

$$\operatorname{Nuc}_{ab}(S)^\times \to \mathfrak{Z}(S, G_{ab})|_{ab}.$$

Proof. If $a = 0$ then $G_{ab} = x_b - x_0$ and the identity map provides the isomorphism. Otherwise $0 < a$, so $G_{ab} = x_b x_a - 1$. It follows that if $(\omega_a, \omega_b) \in \mathfrak{Z}(S, G_{ab})|_{ab}$ then $(\omega_a^{-1}, \omega_b) \in \mathfrak{Z}(S, x_b - x_a)$. □

By composing the inclusion $\operatorname{Nuc}_{ab}(S)^\times \to \mathfrak{Z}(S, G_{ab})$ with $\mathfrak{Z}(S, G_{ab}) \hookrightarrow \mathfrak{Z}(S, G)$ in the diagram of Fact 2.6, we can replace $\coprod$ with $\bigoplus$ in the category of rings. Then, restricting to groups of units, we obtain the following.

Fact 2.9. *There is an exact sequence*

$$\operatorname{Nuc}(S)^\times \xrightarrow{\Gamma_2} \operatorname{Aut}(S) \xrightarrow{\Gamma_1} \prod_{A \in \binom{[\daleth]}{2}} \operatorname{Aut}(S)|_A.$$

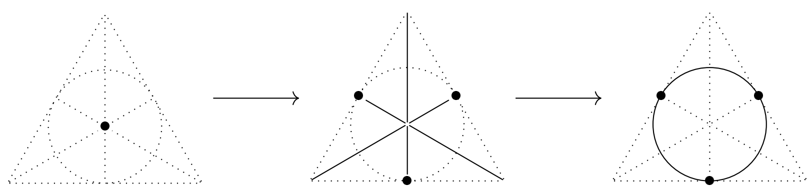
Remark 2.10. For $0 < a < b \leqslant \daleth$, $\mathfrak{Z}(S, x_a + x_b) = \operatorname{Der}_{ab}(S)$ is naturally isomorphic to $\mathfrak{Z}(S, x_a - x_b) = \operatorname{Nuc}_{ab}(S)$ as vector spaces. This implicit isomorphism explains how nuclei appear in Theorem A instead of derivations.

Hereafter, we treat all products as coproducts, and use superscripts ω^A, for $A \subseteq [\daleth]$, to record the factor from which an operator is taken. In this way we obtain explicit (additive) notation for the functions Λ_2^k:

$$\Lambda_2^k\left((\omega_a^A : a \in A) : A \in \binom{[\daleth]}{k}\right) = \sum_{A \in \binom{[\daleth]}{k}} (\epsilon_{\bar{A}}, \omega_A^A). \tag{2.11}$$

Now our goal is to extend the sequence to an exact sequence ending in $\{\epsilon\}$.

2.5. **Summary.** It is possible to state the exact sequences between the algebras and $\mathrm{Cen}_k(S)$ and $\mathrm{Nuc}(S)$ by explicit formulas, and these will be given below. As noted, however, these maps are made for the convenience of working with associative algebras, when in fact that change in categories is unnatural. For instance, Fact 2.9 depends on a twisting of $\mathrm{Nuc}(S)$, and Remark 2.10 is a similar twisting to turn an associative ring into a Lie algebra. Indeed, $\mathrm{Nuc}(S)$ and $\mathrm{Cen}(S)$ only become the rings we seek *after* we restrict the operator sets $\mathfrak{Z}(S, P_A)$ to $\prod_{a\in A} \mathrm{End}(U_a)$. Later we reveal an exponential number of seemingly arbitrary choices in signs that make the sequences. Even so, as we hope we have demonstrated in this section, there is a clear, canonical picture being interpreted through these choices. The following toy example provides a nice illustration of the main ideas.



$\mathfrak{Z}(S, \{x_a - x_b | 0 \leqslant a < b \leqslant 2\})$ $\mathfrak{Z}(S, x_a - x_b)$ $\mathfrak{Z}(S, x_2 + x_1 - x_0)$

FIGURE 2.1. Geometric picture of the operator sets for the exact sequences when $\daleth = 2$.

Defining $\langle t| : \mathbb{K} \times \mathbb{K} \rightarrowtail \mathbb{K}$ with $\langle t|u_2, u_1\rangle = u_2 u_1$, we have

$$\begin{aligned} \mathrm{Der}(t) &= \mathfrak{Z}(t, x_2 + x_1 - x_0) = \{(\lambda_2, \lambda_1, \lambda_2 + \lambda_1)\} \cong \mathbb{K}^2 \\ \mathfrak{Z}(t, x_a - x_b) &= \{(\lambda_2, \lambda_1, \lambda_0) \mid \lambda_a = \lambda_b\} \cong \mathbb{K}^2 \\ \mathrm{Nuc}_{ab}(t) &= \mathfrak{Z}(t, x_a - x_b)|_{ab} = \{(\lambda, \lambda) \in \mathbb{K} \times \mathbb{K}\} \cong \mathbb{K} \\ \mathrm{Cen}(t) &= \{(\lambda, \lambda, \lambda)\} \cong \mathbb{K}. \end{aligned} \tag{2.12}$$

Figure 2.1 illustrates the sets in the projective space $\mathbb{P}^2$. One can see from this figure and the data in (2.12) that the nuclei as rings are not the same as the operator sets, because of the restriction. Furthermore, the centroid is not contained, via inclusion, in the nuclei, but it is a subset of the full operator sets $\mathfrak{Z}(t, x_a - x_b)$. Similarly, the nuclei are not, in general, subsets of derivations. However, the full operator sets $\mathfrak{Z}(t, x_a - x_b)$ can be adjusted naturally to $\mathfrak{Z}(t, x_a + x_b)$, and those intersect nontrivially with $\mathrm{Der}(t)$ from the embedding we described. (In our picture because $|\mathbb{K}| = 2$, where $-1 = +1$, we do not see the distinction.)

For $\daleth = 2$ the sequence is described explicitly as follows. Fix a projective line $\mathfrak{h}$ in $\mathbb{P}^2$. Also fix points $\mathfrak{n}_{ab}$, $0 \leqslant a < b \leqslant 2$ in general position on $\mathfrak{h}$, and a point $\mathfrak{c}$ not on $\mathfrak{h}$. For example, if we use $\mathfrak{n}_{10}$, $\mathfrak{n}_{21}$, and $\mathfrak{c}$ to coordinatize the (affine) plane then $\mathfrak{h}$ is given by a formula of the form $\lambda_2 x_2 + \lambda_1 x_1 + \lambda_0 x_0 = 0$ with each $\lambda_a \neq 0$.

This parametrizes an operator set $\mathfrak{Z}(S, \lambda_2 x_2 + \lambda_1 x_1 + \lambda_0 x_0)$ to which we attach a Lie algebra structure (though now the natural Lie brackets are weighted by the coefficients λ_a). We likewise use the lines through each $\mathfrak{n}_{ab}$ and $\mathfrak{c}$ to define the sets of nucleus type $\mathfrak{Z}(S, \mu_a^{ab} x_a - \mu_b^{ab} x_b)$, and the centroid lies in their intersection. After possibly deforming the scalars of the nuclei, their operator sets intersect the derivation set at the points $\mathfrak{n}_{ab}$. We obtain the desired embedding of $\mathrm{Cen}(S)$ into $\mathrm{Nuc}(S)$ with cokernel embedded into $\mathrm{Der}(S)$.

All this occurs in generic terms and can be reasoned for nonlinear structures such as the operator sets of groups. Our specific interest in derivations and automorphisms are just two natural demonstrations of an otherwise general technique.

3. Exact sequences of groups and algebras

We now specialize our discussion from Section 2 to the sequences in Theorems A and B. With our notation, we consider the case $k = 2$, namely

$$\Lambda_2^2 \colon \bigoplus_{A \in \binom{[\daleth]}{2}} \mathfrak{Z}(S, P_A)|_A \to \mathfrak{Z}(S, P).$$

We also restrict to $P \in \{D, G\}$; in fact, we just consider $P = D$ and observe that the case $P = G$ can be handled in a similar manner by working just with invertible operators. The function Λ_2^2 will now preserve further structure: it will be a Lie algebra homomorphism (or group homomorphism in the $P = G$ case). Recall from Fact 2.9 and Remark 2.10, we must first adjust Λ_2^2 by twisting.

To accomplish this, we define an auxiliary function σ that takes as input a pair of subsets of $[\daleth]$ differing by one element, and returns a value in $\{-1, 1\}$. Recall that subsets of $[\daleth]$ of size 1 are written without $\{\}$. For $A = \{a, b\}$, put $C_A = x_a - x_b$, and define $\Upsilon^2 \colon \bigoplus_{A \in \binom{[\daleth]}{2}} \mathfrak{Z}(S, C_A)|_A \to \mathfrak{Z}(S, P)$ by

$$\bigoplus_{A \in \binom{[\daleth]}{2}} (\omega_a^A : a \in A) \mapsto \left(\sum_{b \in [\daleth] - a} \sigma(a, a \cup b) \cdot \omega_a^{a \cup b} : a \in [\daleth] \right).$$

For $A \subseteq [\daleth]$ of size at least two, let $C_A = \left\{ C_B : B \in \binom{A}{2} \right\}$. For $3 \leqslant k \leqslant \daleth + 1$, if $A \in \binom{[\daleth]}{k}$, from (1.7) we have $\mathfrak{Z}(S, C_A)|_A = \mathrm{Cen}_A(S)$, so

$$\bigoplus_{A \in \binom{[\daleth]}{k}} \mathfrak{Z}(S, C_A)|_A = \mathrm{Cen}_k(S).$$

For $2 \leqslant k \leqslant \daleth$, define $\Upsilon^{k+1} : \bigoplus_{A \in \binom{[\daleth]}{k+1}} \mathfrak{Z}(S, C_A)|_A \to \bigoplus_{B \in \binom{[\daleth]}{k}} \mathfrak{Z}(S, C_B)|_B$ by

$$\bigoplus_{A \in \binom{[\daleth]}{k+1}} (\omega_a^A : a \in A) \mapsto \bigoplus_{B \in \binom{[\daleth]}{k}} \left(\sum_{a \notin B} \sigma(B, B \cup a) \cdot \omega_b^{B \cup a} : b \in B \right).$$

It remains to determine the conditions on σ that ensure the functions Υ^k are well-defined and exact. The next result handles the former requirement.

Lemma 3.1. *For $3 \leqslant k \leqslant \daleth + 1$, the maps Υ^k are well-defined. The map Υ^2 is well-defined if, and only if,*

$$\sigma(a, a \cup b) = \begin{cases} \sigma(b, a \cup b) & 0 = a < b \leqslant \daleth, \\ -\sigma(b, a \cup b) & 0 < a < b \leqslant \daleth. \end{cases} \tag{3.2}$$

Proof. For $A = \{a,b\}$, $\left(\epsilon_{\bar{A}}, \sigma(a,A)\cdot\omega_a^A, \sigma(b,A)\cdot\omega_b^A\right) \in \mathfrak{Z}(S,P)$ if, and only if, (3.2) holds (Fact 2.6 and Lemma 2.8). For $A \subseteq [\vav]$ of size $k \geqslant 3$, if $\omega = (\omega_a^A : a \in A) \in \mathfrak{Z}(S, C_A)|_A$, then

$$\begin{aligned}\Upsilon^k(\omega) &= \bigoplus_{a\in A}\left(\sigma(A-a,A)\cdot\omega_b^A : b \in A-a\right)\\ &= \bigoplus_{a\in A}\sigma(A-a,A)\cdot\left(\omega_b^A : b\in A-a\right).\end{aligned}$$

Thus, each summand is contained in $\mathfrak{Z}(S, C_{A-a})|_{A-a}$. □

For the next two lemmas, we assume that σ satisfies (3.2) so that the maps Υ^k are well-defined for $2 \leqslant k \leqslant \vav + 1$.

Lemma 3.3. *Let S be a fully nondegenerate tensor space. For all $3 \leqslant k \leqslant \vav + 1$, Υ^k is a homomorphism, and Υ^2 is a homomorphism if, and only if,*

$$(\forall 0 \leqslant a < b \leqslant \vav) \qquad\qquad \sigma(a, a\cup b) = \begin{cases} 1 & a = 0,\\ -1 & a > 0.\end{cases}$$

Proof. If S is fully nondegenerate, then for $3 \leqslant k \leqslant \vav + 1$ and $A \in \binom{[\vav]}{k}$, $\mathfrak{Z}(S, P_A)|_A$ is a commutative ring, and so it has trivial Lie bracket and abelian unit group.

We may therefore assume $k = 2$. We give a proof for derivations (where $P = D$); by Lemma 2.8 the proof for autotopisms (where $P = G$) is similar. First, consider $A = \{0, b\}$. By Lemma 3.1, $\sigma(0,A) = \sigma(b,A)$. Recalling that $\mathrm{Nuc}_{0b}(S) \subseteq \mathrm{End}(U_b) \times \mathrm{End}(U_0)$, the map Υ^2_A is a homomorphism if, and only if, $\sigma(0,A) = \sigma(0,A)^2$. Next, if $0 < a < b$, then $\mathrm{Nuc}_{ab}(S) \subseteq \mathrm{End}(U_a)^{\mathrm{op}} \times \mathrm{End}(U_b)$. As the a-coordinate is contained in the opposite ring, Υ^2_A is a homomorphism if, and only if, $\sigma(a,A) = -\sigma(a,A)^2$. □

The notation for the calculations required to prove the exactness of these maps is simplified if, for $C \subset [\vav]$ of order at most $\vav - 1$ and $a, b \in \bar{C}$, we set

$$\tau(C,a,b) = \sigma(C, C\cup a)\sigma(C\cup a, C\cup\{a,b\}) + \sigma(C, C\cup b)\sigma(C\cup b, C\cup\{a,b\}). \tag{3.4}$$

The next result establishes the properties of σ needed to ensure that the maps Υ^* form a chain complex.

Lemma 3.5. *Fix $2 \leqslant k \leqslant \vav$. Then $\Upsilon^k \circ \Upsilon^{k+1} = \epsilon$ if, and only if, for all $C \in \binom{[\vav]}{k-1}$ and distinct $a, b \in \bar{C}$, $\tau(C,a,b) = 0$.*

Proof. For $2 \leqslant k \leqslant \vav$,

$$\begin{aligned}&\Upsilon^k \circ \Upsilon^{k+1}\left(\bigoplus_{A\in\binom{[\vav]}{k+1}} (\omega_a^A : a \in A)\right)\\ &\qquad = \bigoplus_{C\in\binom{[\vav]}{k-1}}\left(\sum_{b\notin C}\sigma(C, C\cup b)\sum_{a\notin C\cup b}\sigma(C\cup b, C\cup\{a,b\})\omega_c^{C\cup\{a,b\}} : c \in C\right)\\ &\qquad = \bigoplus_{C\in\binom{[\vav]}{k-1}}\left(\sum_{\{a,b\}\in\binom{[\vav]-C}{2}}\tau(C,a,b)\omega_c^{C\cup\{a,b\}} : c\in C\right).\end{aligned}$$ □

We next show that the conditions in Lemma 3.5—which are clearly necessary for exactness of Υ^*—are also sufficient.

Lemma 3.6. *Fix* $2 \leqslant k \leqslant \daleth$. *If* $\Upsilon^k \circ \Upsilon^{k+1} = \epsilon$, *then* $\ker(\Upsilon^k) = \mathrm{im}(\Upsilon^{k+1})$.

Proof. We will express Υ^k as a matrix with entries in $\{-1, 0, 1\}$. The nonzero entries are determined by the function σ. Our approach is then to derive a suitable transition matrix M_k and study instead $\Upsilon^k M_{k+1} M_{k+1}^{-1} \Upsilon^{k+1}$. First, we arrange the components of $\bigoplus_{A \in \binom{[\daleth]}{k+1}} \mathfrak{Z}(S, C_A)|_A$ so that the first summand to be over $[k]$, and thus collect together those operators that act on the same term in the frame:

$$\bigoplus_{A \in \binom{[\daleth]}{k+1}} (\omega_a^A : a \in A) \mapsto \bigoplus_{a \in [\daleth]} \left(\omega_a^{B \cup a} : B \in \binom{[\daleth] - a}{k} \right).$$

Reordering Υ^2 into these coordinates, we rewrite Υ^{k+1} in terms of its restrictions Υ_a^{k+1} to each $\mathrm{End}(U_a)$, as follows:

$$\Upsilon^{k+1} \left(\bigoplus_{A \in \binom{[\daleth]}{k+1}} (\omega_a^A : a \in A) \right) = \bigoplus_{a \in [\daleth]} \Upsilon_a^{k+1} \left(\omega_a^{B \cup a} : B \in \binom{[\daleth] - a}{k} \right).$$

Secondly, we use a matrix to describe each Υ_a^{k+1} as it naturally leads to the concept of echelonizing, which will identify the image of each map. Rather than work with a block diagonal matrix $\Upsilon^{k+1} = \bigoplus_a \Upsilon_a^{k+1}$, we instead fix $a \in [\daleth]$ and focus on the matrix for just that fixed coordinate. Thus, for each k we define a $\{-1, 0, 1\}$-valued matrix M_{k+1} whose rows range over $\binom{[\daleth]-a}{k}$ and whose columns range over $\binom{[\daleth]-a}{k-1}$. Fixing a total order (e.g. lexicographic) on the subsets of $[\daleth] - a$, define

$$\left[M_{k+1}^{(a)} \right]_{AB} = \begin{cases} \sigma(B, A) & B \subset A, \\ 0 & B \not\subset A. \end{cases}$$

For $b = \min([\daleth] - a)$, we observe the following partition of $M_{k+1}^{(a)}$:

$$M_{k+1}^{(a)} = \begin{array}{c} \\ b \in A \\ b \notin A \end{array} \begin{array}{c} \begin{array}{cc} b \in B & b \notin B \end{array} \\ \left[\begin{array}{c|c} Y_{k+1} & Z_{k+1} \\ \hline 0 & X_{k+1} \end{array} \right] \end{array}, \qquad Z_{k+1} = \bigoplus_{a \in A - B} \sigma(B, B \cup a).$$

The block of 0's in the lower left follows from the fact that those rows and columns are indexed by subsets A and B for which $B \not\subset A$. Similarly in the upper right block $B \subset A$ at exactly the row $A = B \cup a$ and column B, for our fixed a. As $\sigma(B, B \cup a) \in \{-1, 1\}$, it follows that Z_{k+1} is diagonal, invertible, and of order at most 2. Letting $Z_1 = 1$ for the base case, we now claim that for each k,

$$Y_{k+1} = -Z_{k+1} X_k Z_k. \tag{3.7}$$

Indeed, as Z_{k+1} is an involution we can rewrite this identity equivalently as

$$Z_{k+1} X_k + Y_{k+1} Z_k = 0. \tag{3.8}$$

Focusing just on non-zero coordinates, condition (3.8) translates precisely to the condition $\tau(C, c, d) = 0$, as in in (3.4). Thus, by Lemma 3.5, equation (3.7) holds.

Having identified the structure of the matrices, for each $a \in [\daleth]$ we can echelonize the coordinates of the transform Υ_a^{k+1} to identify the symbolic rank of each matrix. This will ensure that the image equals the kernel. To simplify notation for inverses, we put $N_k = M_k^{(a)}$, and compute

$$\begin{aligned}
N_2^{-1}\Upsilon_a^2 &= \begin{bmatrix} Z_2 & 0 \\ -X_2 Z_2 & I \end{bmatrix} \begin{bmatrix} Z_2 \\ X_2 \end{bmatrix} && = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&\vdots && \vdots \\
N_{k+1}^{-1}\Upsilon_a^{k+1} N_k &= \begin{bmatrix} Z_{k+1} & 0 \\ -X_{k+1} Z_{k+1} & I \end{bmatrix} \begin{bmatrix} -Z_{k+1} X_k Z_k & Z_{k+1} \\ 0 & X_{k+1} \end{bmatrix} \begin{bmatrix} Z_k & 0 \\ X_k & I \end{bmatrix} && = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \\
&\vdots && \vdots \\
\Upsilon_a^{\daleth+1} N_\daleth &= \begin{bmatrix} Z_\daleth X_{\daleth-1} Z_{\daleth-1} & Z_\daleth \end{bmatrix} \begin{bmatrix} Z_\daleth & 0 \\ Z_\daleth X_{\daleth-1} Z_{\daleth-1} & Z_\daleth \end{bmatrix} && = \begin{bmatrix} 0 & 1 \end{bmatrix}
\end{aligned}$$

Evidently the image (column span) of each matrix on the right is the kernel (right null space) of the one below it. Since each row-column operation is carried out by unimodular transform (a permutation or transvection with ± 1-values) it is possible to perform these operations symbolically on our coordinates. Thus, elements in the kernel of Υ_a^k can be used to write elements in the image of Υ_a^{k+1}. □

3.1. **Conditions on σ.** To recap, three properties are required of σ that ensure the homomorphisms Υ^* form an exact sequence. First, by Lemma 3.1, we require

$$\sigma(a, a \cup b) = \begin{cases} \sigma(b, a \cup b) & 0 = a < b \leqslant \daleth, \\ -\sigma(b, a \cup b) & 0 < a < b \leqslant \daleth, \end{cases}$$

so that the maps are well-defined. Secondly, from Lemma 3.3 we require

$$\sigma(a, a \cup b) = \begin{cases} 1 & 0 = a < b \leqslant \daleth, \\ -1 & 0 < a < b \leqslant \daleth, \end{cases}$$

to ensure the homomorphism property. Finally, by Lemmas 3.5 and 3.6 we require that for all $C \subset [\daleth]$ with order at most $\daleth - 1$, and for all distinct $a, b \notin C$,

$$\sigma(C, C \cup a)\sigma(C \cup a, C \cup \{a,b\}) + \sigma(C, C \cup b)\sigma(C \cup b, C \cup \{a,b\}) = 0.$$

4. Directed graphs

The goal of this section is to prove the existence of a function σ satisfying the conditions laid out in Section 3.1. This is accomplished by means of a directed graph $\mathcal{G}_\daleth$ whose vertices are subsets of $[\daleth]$. Two vertices $A, B \subseteq [\daleth]$ are adjacent if there exists $b \notin A$ such that $A \cup b = B$. Our objective is to define an *orientation* on $\mathcal{G}_\daleth$, namely to assign a direction to each edge in $\mathcal{G}_\daleth$ that encodes the nonzero values of σ. A directed edge $C \cup a \to C$ can be thought of as "down" in the underlying poset and carries a value of 1, while $C \to C \cup a$ is "up" and carries the value -1.

To state the key result, we introduce some convenient terminology. We refer to an edge from $A \subseteq [\daleth - 1]$ to $A \cup \daleth$ as a *controlling edge*. For distinct $a, b \in [\daleth]$ and $C \subseteq [\daleth] - \{a, b\}$, denote by $\mathcal{D}(C, a, b)$ the subgraph of $\mathcal{G}_\daleth$ induced on the four vertices labeled by C, $C \cup a$, $C \cup b$, and $C \cup \{a, b\}$. We refer to subgraphs $\mathcal{D}(C, a, b)$ as *diamonds* of $\mathcal{G}_\daleth$. We say an orientation on $\mathcal{G}_\daleth$ is *oddly acyclic* if every diamond of $\mathcal{G}_\daleth$ is acyclic with a path of length 3. For consistency, we say that every orientation on $\mathcal{G}_0$ is (vacuously) oddly acyclic.

Our first result collects the diamonds of $\mathcal{G}_\daleth$ into three classes. For $A \subseteq [\daleth]$, we say a diamond $\mathcal{D}$ is contained in 2^A, the power set of A, if the vertex labels of $\mathcal{D}$ are contained in A.

Lemma 4.1. *For $\daleth \geqslant 1$, every diamond of $\mathcal{G}_\daleth$ is either contained in $2^{[\daleth-1]}$, contained in $2^{[\daleth]} - 2^{[\daleth-1]}$, or contains exactly two controlling edges. This partitions the set of diamonds of $\mathcal{G}_\daleth$.*

Proof. No diamond in $2^{[\daleth-1]}$ or $2^{[\daleth]} - 2^{[\daleth-1]}$ contains a controlling edge, so the three sets are disjoint. Suppose $\mathcal{D}$ is a diamond not contained in $2^{[\daleth-1]}$ or $2^{[\daleth]} - 2^{[\daleth-1]}$. Because $2^{[\daleth]}$ and $2^{[\daleth]} - 2^{[\daleth-1]}$ are subset-closed, it follows that $\mathcal{D}$ has two vertices with labels in $2^{[\daleth]}$ and two vertices with labels in $2^{[\daleth]} - 2^{[\daleth-1]}$. □

We dedicate the key result in this section to the inspiration for its proof:

Lemma 4.2 (Rihanna's Lemma)**.** *Suppose $\daleth \geqslant 1$. For every oddly acyclic orientation on the subgraph $\mathcal{G}_{\daleth-1}$ of $\mathcal{G}_\daleth$, and for every orientation on the controlling edges of $\mathcal{G}_\daleth$, there exists a unique induced oddly acyclic orientation for $\mathcal{G}_\daleth$.*

Proof. Think of diamonds contained in $2^{[\daleth-1]}$ as *green diamonds*, and of those contained in $2^{[\daleth]} - 2^{[\daleth-1]}$ as *yellow diamonds*. Call a diamond *controlling* if it contains two controlling edges. Observe that every non-controlling edge of $\mathcal{G}_\daleth$ lies in a unique controlling diamond.

We construct the induced orientation from the orientations on $\mathcal{G}_{\daleth-1}$ and the controlling edges of $\mathcal{G}_\daleth$ as follows. For every $a \in [\daleth-1]$ and $C \subseteq [\daleth-1] - a$, the (yellow) edge y incident to $C \cup \{a, \daleth\}$ and $C \cup \{\daleth\}$ lies in a unique controlling diamond $\mathcal{D}(C, a, \daleth)$. As a scholium to Lemma 4.1, we observe that the orientation of exactly three edges of $\mathcal{D}(C, a, \daleth)$ is determined by the oddly acyclic orientation on $\mathcal{G}_{\daleth-1}$ and the orientation on the controlling edges of $\mathcal{G}_\daleth$. Thus, there is a unique choice of orientation of y such that $\mathcal{D}(C, a, \daleth)$ is oddly acyclic. Impose this orientation upon all yellow edges, and consider the resulting directed graph $\mathcal{G}_\daleth$.

By Lemma 4.1 we must show that every yellow diamond is oddly acyclic. Such a diamond has the form $\mathcal{D} = \mathcal{D}(C, a, b)$, where $\{\daleth\} \subseteq C \subseteq [\daleth]$ and $a, b \in [\daleth-1]$ with $a \neq b$. Let $\mathcal{D}' = \mathcal{D}(C - \daleth, a, b)$, a green diamond associated to $\mathcal{D}$. There are exactly four controlling edges incident to both $\mathcal{D}$ and $\mathcal{D}'$. Two controlling edges incident to the same edges, y in $\mathcal{D}$ and g in $\mathcal{D}'$, point in the same direction if, and only if, the edges y and g point in opposite directions. This implies that $\mathcal{D}$ is oddly acyclic, since by hypothesis $\mathcal{D}'$ is oddly acyclic. The result now follows. □

Rihanna's Lemma is illustrated in Figure 4.1 for $\daleth = 2$. The orientations for the green diamond and the controlling edges have been given, and by Rihanna's Lemma there is a unique choice of orientation for the yellow diamond so that $\mathcal{G}_2$ is oddly acyclic.

The final result of this section establishes the existence of a suitable function σ to use in our exact sequences. First some additional terminology: an orientation on $\mathcal{G}_\daleth$ is *positively swapped* if the subgraph induced on the vertices labeled $\{a, b\}$, $\{a\}$, and $\{b\}$ is a path of length 2 if, and only if, $0 < a < b \leqslant \daleth$.

Lemma 4.3. *For every $\daleth \geqslant 1$, there exists an oddly acyclic and positively swapped orientation for $\mathcal{G}_\daleth$ such that for every $A \in \binom{[\daleth]}{2}$, there is a directed edge from A to* $\max(A)$.

Proof. We prove this by induction. From Rihanna's Lemma, there are 8 oddly acyclic orientations for $\mathcal{G}_1$, exactly 4 are of those positively swapped. Of the 4 orientations, exactly 2 have a directed edge from $\{0, 1\}$ to $\{1\}$.

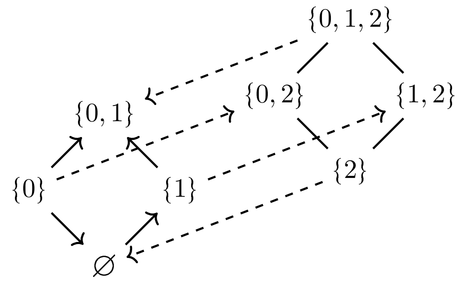


FIGURE 4.1. We illustrate $\mathcal{G}_2$, with the "green" diamond on the lower left and "yellow" diamond on the upper right. The controlling edges (dashed) connecting the two colored diamonds.

Now suppose the subgraph $\mathcal{G}_{\daleth-1}$ is oddly acyclic and positively swapped such that for every $A \in \binom{[\daleth-1]}{2}$, there is a directed edge from A to $\max(A)$. The only choice we impose to get positively swapped is that the controlling edge between $\{\daleth\}$ and $\varnothing$ points in a direction such that the subgraph on $\{0\}$, $\{\daleth\}$, and $\varnothing$ is a path of length 2. By induction, the subgraph on $\{0\}$, $\{a\}$, and $\varnothing$ is a path of length 2, for all $0 < a < \daleth$. Therefore, regardless of the choice of orientation for the remaining controlling edges, the resulting orientation from Rihanna's Lemma will be positively swapped. To ensure the last property, we choose the orientation for the controlling edges so that the edge points from $\{a, \daleth\}$ to $\{\daleth\}$, for all $a \in [\daleth-1]$. □

5. PROOF OF THEOREMS A AND B

Both theorems follow easily from our work in Sections 3 and 4.

Lemma 5.1. *For $\daleth \geqslant 1$, there exists a function σ such that*

(*i*) *for all $A \subset [\daleth]$ and $b \notin A$,*

$$\sigma(A, A \cup b) = \pm 1,$$

(*ii*) *for all $0 \leqslant a < b \leqslant \daleth$,*

$$\sigma(a, a \cup b) = \begin{cases} \sigma(b, a \cup b) & 0 = a, \\ -\sigma(b, a \cup b) & 0 < a, \end{cases} \qquad \sigma(a, a \cup b) = \begin{cases} 1 & 0 = a, \\ -1 & 0 < a, \end{cases}$$

(*iii*) *for all $C \in \binom{[\daleth]}{k-1}$ and distinct $a, b \notin C$,*

$$\sigma(C, C \cup a)\sigma(C \cup a, C \cup \{a, b\}) + \sigma(C, C \cup b)\sigma(C \cup b, C \cup \{a, b\}) = 0.$$

Proof. Translating the directions on edges in an oriented graph $\mathcal{G}_\daleth$ to values ± 1—as described at the start of Section 4—each such graph encodes some function σ with the correct domain and range in property (i). By Lemma 4.3, the orientation on $\mathcal{G}_\daleth$ can be chosen so that properties (ii) and (iii) hold. □

Proof of Theorems A and B. By Lemmas 3.1, 3.3, 3.6, 5.1, and full nondegeneracy of S, the homomorphisms Υ^k form an exact sequence. Using Fact 2.3, if $P \in \{D, G\}$, then the zero set $\mathfrak{Z}(S, P)$ is either $\mathrm{Der}(S)$ or $\mathrm{Aut}(S)$. In the case $P = G$, apply Lemma 2.8 to obtain the sequence in Theorem B from the zero-sets. □

6. Applications & Examples

The techniques developed in Sections 2 and 3 to prove Theorems A and B being rather new, it might not be clear to the reader how they can be used to study tensors, nor how they might be applied in more general settings. We therefore devote this final section to a range of examples and applications that illustrate the scope and potential of these new tools.

6.1. **Entangled quantum states.** In [DVC], Dür-Vidal-Cirac considered two maximally entangled quantum states on 3 qubits, and demonstrated they are inequivalent. Here, we provide an elementary verification of their discovery using our exact sequences. Let $\mathbb{H}$ be an 8-dimensional Hilbert space with $\langle\cdot| : \mathbb{H} \to \mathbb{C}^2 \oslash \mathbb{C}^2 \oslash \mathbb{C}^2$ (more commonly represented as $\mathbb{C}\oslash(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2)$, the dual space to $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$). In the usual convention, $\mathbb{C}^2 = \mathbb{C}\langle 0| \oplus \mathbb{C}\langle 1|$, so a basis for $\mathbb{H}$ is $\langle abc|$ with $a, b, c \in \{0, 1\}$. The *Greenberger-Horne-Zeilinger* state is defined

$$\langle GHZ| = \frac{\sqrt{2}}{2}(\langle 000| + \langle 111|),$$

while the state W from [DVC] is

$$\langle W| = \frac{\sqrt{3}}{3}(\langle 100| + \langle 010| + \langle 001|).$$

An elementary calculation reveals that $\mathrm{Cen}(GHZ) \cong \mathbb{C}^2$, and the sequence is

$$0 \longrightarrow \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \oplus \mathbb{C}^2 \longrightarrow 0.$$

On the other hand, $\mathrm{Cen}(W) \cong \mathbb{C}[x]/(x^2)$. Even though the centroid is also 2-dimensional, the algebra is fundamentally different—in particular, it has a nontrivial Jacobson radical. Furthermore, the corresponding sequence for W contains nontrivial outer derivations; with $A = \mathbb{C}[x]/(x^2)$, we have

$$0 \longrightarrow A \longrightarrow A \oplus A \oplus A \longrightarrow \mathbb{C} \oplus A \oplus A \longrightarrow \mathbb{C} \longrightarrow 0.$$

The two states are clearly inequivalent.

6.2. **Composition and matrix products.** This example uses the familiar concept of *tensor contraction* (also known as *hyper-matrix multiplication*), which for convenience we model as a special case of composition in a module category.

Recall in our notation $V \oslash U = \mathrm{Hom}(U, V)$, which is again a module. Hence, composition of functions in the K-module category is a bilinear map

$$\circ : A \oslash B \times B \oslash C \rightarrowtail A \oslash C.$$

If $A = K^a$, $B = K^b$, and $C = K^c$, we can, after fixing bases, identify composition with the *matrix multiplication tensor*

$$\mathbb{M}_{a\times b}(K) \times \mathbb{M}_{b\times c}(K) \rightarrowtail \mathbb{M}_{a\times c}(K).$$

The latter has been studied extensively, and its derivations and automorphisms are known. Thus, it is a good example to illustrate our methods, as they provide another means to see the structure.

Within composition we have three self-evident contributions to the nuclei:

$$\mathrm{Nuc}_{20}(\circ) \cong \mathrm{End}(A), \qquad \mathrm{Nuc}_{21}(\circ) \cong \mathrm{End}(B), \qquad \mathrm{Nuc}_{10}(\circ) \cong \mathrm{End}(C).$$

Therefore, while B does not occur in the codomain, its influence in the middle of the domain can be identified by the nuclei. This may not seem surprising given how we introduced the product, but such a tensor could be given as a black-box. Then the product would take the form $K^{ab} \times K^{bc} \rightarrowtail K^{ac}$, and that is a completely ambiguous decomposition. Let us consider a specific example.

Example 6.1. If $t\colon \mathbb{M}_{2\times 3}(\mathbb{C}) \times \mathbb{M}_{3\times 4}(\mathbb{C}) \rightarrowtail \mathbb{M}_{2\times 4}(\mathbb{C})$, then

$$\mathrm{Cen}_3(t) \cong \mathbb{C}, \qquad \mathrm{Nuc}(t) \cong \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_3(\mathbb{C}) \oplus \mathbb{M}_4(\mathbb{C}).$$

Therefore, the sequences in Theorem A and Theorem B have the form

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{M}_2 \oplus \mathbb{M}_3 \oplus \mathbb{M}_4 \longrightarrow (\mathfrak{gl}_2 \oplus \mathfrak{gl}_3 \oplus \mathfrak{gl}_4)/\mathbb{C} \longrightarrow 0,$$
$$1 \longrightarrow \mathbb{C}^{\times} \longrightarrow \mathrm{GL}_2 \times \mathrm{GL}_3 \times \mathrm{GL}_4 \longrightarrow \mathrm{SL}_3 \rtimes (\mathrm{GL}_2 \times \mathrm{GL}_4) \longrightarrow 1.$$

In other words, all autotopisms of tensors given by matrix multiplication are "inner," in the sense that they are realized as the groups of units of the various nuclei.

6.3. **Non-associative tensor decompositions.** Tensor contraction is expressible in other ways, partly because it relates to our familiar associative matrix multiplication. However, tensors can also be constructed in non-associative ways; one need only consider products of Lie algebras to witness such cases. Our next examples demonstrate how we detect nonassociative components of a tensor.

Example 6.2. Let K be a field such that $2K = K$. Let $\langle t|\colon K^4 \times K^4 \times K^4 \rightarrowtail \bigwedge^3 K^4$ be given by the exterior cube of K^4. Also, let $\langle s| : K \times \mathfrak{sl}_2 \times \mathfrak{sl}_2 \rightarrowtail \mathfrak{sl}_2$ be the tensor that maps (k, X, Y) to $k[X, Y]$. Define the tensor product (over K) of K-tensors t and s as $\langle t \otimes s|\colon (K^4 \otimes K) \times (K^4 \otimes \mathfrak{sl}_2) \times (K^4 \otimes \mathfrak{sl}_2) \rightarrowtail K^4 \otimes \mathfrak{sl}_2$, where

$$\langle t \otimes s|u \otimes v\rangle = \langle t|u\rangle \otimes \langle s|v\rangle.$$

A calculation shows that $\mathrm{Der}(t) \cong K^2 \oplus \mathfrak{gl}_4(K)$. For all other $A \subset [3]$, with $|A| \geqslant 2$, $\mathrm{Der}_A(t) \cong K^{|A|-1}$. For s, on the other hand, $\mathrm{Der}_{\{0,1,2\}}(s) \cong K \oplus \mathfrak{gl}_3(K)$ and $\mathrm{Der}(s) \cong K \oplus \mathrm{Der}_{\{0,1,2\}}(s)$. For all other $A \subset [3]$, $\mathrm{Der}_A(s) \cong K^{|A|-1}$. These derivation algebras are detected by the sequence in Theorem A for $t \otimes s$. In fact, there are no larger algebras:

$$0 \longrightarrow K \longrightarrow K^4 \longrightarrow K^6 \longrightarrow K \oplus \mathfrak{gl}_3 \oplus \mathfrak{gl}_4 \longrightarrow \mathfrak{sl}_3 \oplus \mathfrak{sl}_4 \longrightarrow 0.$$

Example 6.3. Let K be a field with degree 2 and 3 extensions denoted by E and F respectively. Let t and s be the dot products on E^2 and F^2, respectively. We concatenate a 1-dimensional coordinate (over K) to both t and s, making them K-trilinear: $\langle t'| : K \times K^4 \times K^4 \rightarrowtail K^2$ and $\langle s'| : K^6 \times K^6 \times K \rightarrowtail K^3$. Let $r = t' \otimes s'$. Instead of writing out the sequence like in the previous examples, we display the dimension of every algebra over K in Figure 6.1. While r is only K-trilinear (the centroid of r is isomorphic to K), the local centroids detect the fact that r was built from tensors that are *bi*-linear over extensions, namely degree 2 and 3 extensions.

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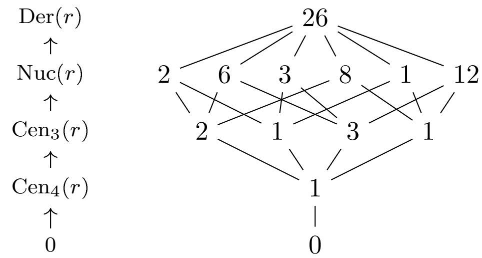

FIGURE 6.1. A graphical description of the sequence in Theorem A. Here, we have separated the direct summands of the terms of the sequence, and we are only displaying their dimensions over K. The sequence starts at the bottom and goes to the top, with the last nontrivial term being the 26-dimensional derivation algebra. The vertical sequence on the left aligns with the dimensions of the direct summands—in lex-least order.

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