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Hall–Littlewood polynomials, affine Schubert series, and lattice enumeration

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Abstract. We introduce multivariate rational generating series called Hall–Littlewood–Schubert (HLS_n) series. They are defined in terms of polynomials related to Hall–Littlewood polynomials and semistandard Young tableaux. We show that HLS_n series provide solutions to a range of enumeration problems upon judicious substitutions of their variables. These include the problem to enumerate sublattices of a p-adic lattice according to the elementary divisor types of their intersections with the members of a complete flag of reference in the ambient lattice. This is an affine analog of the stratification of Grassmannians by Schubert varieties. Other substitutions of HLS_n series yield new formulae for Hecke series and p-adic integrals associated with symplectic p-adic groups, and combinatorially defined quiver representation zeta functions. HLS_n series are q-analogs of Hilbert series of Stanley–Reisner rings associated with posets arising from parabolic quotients of Coxeter groups of type B with the Bruhat order. Special values of coarsened HLS_n series yield analogs of the classical Littlewood identity for the generating functions of Schur polynomials.

Keywords: Bruhat order, Dyck words, functional equations, quiver representation zeta functions, Gelfand–Tsetlin patterns, Hall–Littlewood polynomials, Igusa functions, p-adic integration, Schur polynomials, semistandard Young tableaux, Stanley–Reisner rings, submodule zeta function, symmetric functions, symplectic groups, symplectic Hecke series

Introduction

We offer a unifying framework for a wide variety of counting problems from geometry, number theory, and algebra. To this end we introduce *Hall–Littlewood–Schubert series* HLS_n for $n \in \mathbb{N}$; see Definition 1.2. These are multivariate rational generating functions defined as sums over semistandard Young tableaux (or just tableaux in the sequel), involving polynomials related to Hall–Littlewood polynomials. We show that they specialize, under judicious substitutions of their 2^n variables, to generating series solving various counting problems.

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What makes each of these problems amenable to Hall–Littlewood–Schubert series is that they all factor over natural maps from the set of all finite-index sublattices of a fixed lattice of finite rank n to the infinite set SSYT_n of tableaux with entries from $\{1, \dots, n\}$. In each case, the key to reducing the respective counting problem to HLS_n is to compute and enumerate the fibers of the relevant map. En route we discover connections with further classical objects of algebraic combinatorics, such as Dyck words, the Bruhat order, and Stanley–Reisner rings. Three such instantiations, all related to lattice enumeration, stand out.

(1) Let V be a module over a compact discrete valuation ring $\mathfrak{o}$, free of finite rank n, equipped with a complete flag of isolated submodules $\{0\} = V^{(0)} \subsetneq V^{(1)} \subsetneq V^{(2)} \subsetneq \cdots \subsetneq V^{(n)} = V$. The *affine Schubert series* $\mathsf{affS}^{\mathrm{in}}_{n,\mathfrak{o}}$ introduced in Definition 2.1 enumerates sublattices of finite index in V by the elementary divisors of their intersections with each of the lattices $V^{(i)}$. This may be seen as an affine analog of the classical concept of Schubert varieties, stratifying Grassmannians by the intersection dimensions with a fixed complete flag in the ambient vector space; see [7]. Theorem B asserts that HLS_n specializes to $\mathsf{affS}^{\mathrm{in}}_{n,\mathfrak{o}}$ under a monomial substitution of the variables. Theorem C is a similar result for the affine Schubert series $\mathsf{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}$, enumerating lattices by the elementary divisors of their projections to, rather than intersections with, the members of a complete flag of reference.

(2) *Hecke series* play an important role in algebra and number theory. Theorem E shows that Hall–Littlewood–Schubert series HLS_n specialize to the Hecke series associated with groups of symplectic similitudes over local fields as studied by Macdonald [13, Chapter V]. This leads to new formulae for and new results about these classical series. As byproducts we prove, for instance, conjectures raised in [16, 22].

(3) *Quiver representation zeta functions* enumerate subrepresentations of integral quiver representations; see [11]. Specializations of Hall–Littlewood–Schubert series yield new and explicit formulae for these zeta functions associated with combinatorially defined quiver representations over compact discrete valuation rings; see Theorem F. Our work brings ideas and tools from algebraic combinatorics to bear where previously algebro-geometric methods dominated.

Our explicit formulae show that the generating series associated with these lattice enumeration problems depend only mildly on the local rings over which they are defined. More precisely, they all turn out to be rational functions whose coefficients are polynomials in the residue field cardinalities. This so-called uniformity is reminiscent of the well-known fact that the numbers of rational points of Schubert varieties over finite fields are given by integral polynomials in the cardinalities of these fields.

Additional applications flow from the fact that HLS_n is a Y-analog of the Hilbert series of the Stanley–Reisner ring of a natural simplicial complex. This is the order complex $\Delta(\mathsf{T}_n)$ of the poset $\mathsf{T}_n = 2^{[n]} \setminus \{\varnothing\}$ equipped with the *tableaux order* introduced in Section 6. The poset T_n may be interpreted in terms of the Bruhat order on parabolic quotients of finite Coxeter groups of type B [23, Theorem 1].

We state a general self-reciprocity result for the Hall–Littlewood–Schubert series HLS_n upon inversion of their variables (Theorem A). Through the relevant variable substitutions, self-reciprocity is passed on to the generating series described above, vastly extending the scope of this well-studied symmetry phenomenon. Our proof of Theorem A is facilitated by interpreting HLS_n in terms of $\mathfrak{p}$-adic integrals. Conversely, we give pleasing formulae for well-studied $\mathfrak{p}$-adic integrals associated with symplectic $\mathfrak{p}$-adic groups in terms of Hall–Littlewood–Schubert series.

This abstract is an exposition of some results from our preprint [14].

1 Hall–Littlewood–Schubert series

For a tableau $T = (T_{ij})$ in SSYT_n, write $T = (C_1, \dots, C_\ell)$ to denote the columns of T. For $i, j \in \mathbb{N}$ we define the ***leg set*** of T:

$$\mathrm{Leg}_T^+(i,j) = \begin{cases} C_j \cap [T_{ij},\ T_{i(j+1)}] & \text{if } T_{i(j+1)} \notin C_j, \\ \varnothing & \text{otherwise.} \end{cases}$$

We set $\mathscr{L}_T = \left\{(i,j) \in \mathbb{N}^2 \mid \mathrm{Leg}_T^+(i,j) \neq \varnothing\right\}$.

Definition 1.1. The ***leg polynomial*** associated with $T \in \mathrm{SSYT}_n$ is

$$\Phi_T(Y) = \prod_{(i,j)\in\mathscr{L}_T} \left(1 - Y^{\#\mathrm{Leg}_T^+(i,j)}\right) \in \mathbb{Z}[Y].$$

We introduce further $2^n - 1$ variables $\boldsymbol{X} = (X_C)_{\varnothing \neq C \subseteq [n]}$. We call a tableau ***reduced*** if its columns are pairwise distinct and write rSSYT_n for the finite (!) subset of reduced tableaux of SSYT_n.

Definition 1.2. The ***Hall–Littlewood–Schubert series*** is

$$\mathsf{HLS}_n\left(Y, \boldsymbol{X}\right) = \sum_{T\in\mathrm{rSSYT}_n} \Phi_T(Y) \prod_{C\in T} \frac{X_C}{1 - X_C} \in \mathbb{Z}[Y]\left(\boldsymbol{X}\right).$$

Remark 1.3. The leg polynomial Φ_T coincides with a known polynomial invariant of Gelfand–Tsetlin patterns, written p_A in [6, Theorem 1.1]. We also note that leg sets index the cells contained in the leg of the (i,j)-cell for a suitable partition in Macdonald's terminology; see [13, p. 337]. In [14] we give an interpretation of leg polynomials in terms of statistics on Dyck words.

We define the denominator polynomial $\mathsf{D}_n(\boldsymbol{X}) = \prod_{\varnothing\neq C\subseteq[n]}(1 - X_C) \in \mathbb{Z}[\boldsymbol{X}]$. We then define the numerator polynomial $\mathsf{N}_n(Y, \boldsymbol{X}) \in \mathbb{Z}[Y, \boldsymbol{X}]$ via

$$\mathsf{HLS}_n(Y, \boldsymbol{X}) = \frac{\mathsf{N}_n(Y, \boldsymbol{X})}{\mathsf{D}_n(\boldsymbol{X})}. \tag{1.1}$$

Example 1.4 (HLS_n for $n \leqslant 3$)**.** Given subsets $I_1, I_2, \ldots \subset \mathbb{N}$ we write $\boldsymbol{X}_{I_1|I_2|\ldots} = X_{I_1}X_{I_2}\cdots$. We further simplify the subscripts by displaying only the sets' elements: e.g. we write X_{13} instead of $X_{\{1,3\}}$. For $n \leqslant 3$, we find $\mathsf{N}_1(Y,\boldsymbol{X}) = 1$ and $\mathsf{N}_2(Y,\boldsymbol{X}) = 1 - Y\boldsymbol{X}_{1|2}$, and

$$\begin{aligned}
\mathsf{N}_3(Y,\boldsymbol{X}) &= 1 - \boldsymbol{X}_{1|23} \\
&- Y\left(\boldsymbol{X}_{1|2} + \boldsymbol{X}_{1|3} + \boldsymbol{X}_{2|3} + \boldsymbol{X}_{2|13} + \boldsymbol{X}_{12|13} + \boldsymbol{X}_{12|23} + \boldsymbol{X}_{13|23} + \boldsymbol{X}_{1|2|13|23}\right) \\
&+ Y\left(\boldsymbol{X}_{1|2|3} + \boldsymbol{X}_{1|2|13} + \boldsymbol{X}_{1|2|23} + \boldsymbol{X}_{1|3|23} + \boldsymbol{X}_{1|12|23} + \boldsymbol{X}_{1|13|23} + \boldsymbol{X}_{2|13|23} + \boldsymbol{X}_{12|13|23}\right) \\
&+ Y^2\left(\boldsymbol{X}_{1|2|3} + \boldsymbol{X}_{1|3|12} + \boldsymbol{X}_{2|3|12} + \boldsymbol{X}_{2|3|13} + \boldsymbol{X}_{2|12|13} + \boldsymbol{X}_{3|12|13} + \boldsymbol{X}_{3|12|23} + \boldsymbol{X}_{12|13|23}\right) \\
&- Y^2\left(\boldsymbol{X}_{3|12} + \boldsymbol{X}_{1|2|3|12} + \boldsymbol{X}_{1|2|3|13} + \boldsymbol{X}_{1|2|3|23} + \boldsymbol{X}_{1|3|12|23} + \boldsymbol{X}_{1|12|13|23}\right. \\
&\qquad \left. + \boldsymbol{X}_{2|12|13|23} + \boldsymbol{X}_{3|12|13|23}\right) \\
&- Y^3\left(\boldsymbol{X}_{2|3|12|13} - \boldsymbol{X}_{1|2|3|12|13|23}\right). \qquad \diamondsuit
\end{aligned}$$

Our first main result establishes a general self-reciprocity property for HLS_n.

Theorem A. *We have*

$$\mathsf{HLS}_n(Y^{-1}, \boldsymbol{X}^{-1}) = (-1)^n Y^{-\binom{n}{2}} X_{[n]} \cdot \mathsf{HLS}_n(Y, \boldsymbol{X}).$$

(Self-)Reciprocity results as the one established in Theorem A are ubiquitous, but not universal, phenomena seen in numerous counting problems in algebra, geometry and combinatorics; see, for instance, [2, 15, 25]. As a corollary we obtain reciprocity results for instantiations of HLS_n. One such result is Corollary 4.1, which establishes a functional equation for Fourier transforms of the Hecke series associated with symplectic groups.

We now present the principal applications of Hall–Littlewood–Schubert series to $\mathfrak{p}$-adic lattice enumeration problems as well as some of their combinatorial and topological properties.

2 Affine Schubert series

Enumerating full lattices in $\mathbb{Z}^n$ by their index is a classical problem with a well-known solution. The monograph [12] lists no fewer than five proofs of the following identity:

$$\zeta_{\mathbb{Z}^n}(s) := \sum_{\Lambda \leqslant \mathbb{Z}^n} |\mathbb{Z}^n : \Lambda|^{-s} = \prod_{i=0}^{n-1} \zeta(s-i), \tag{2.1}$$

where the sum runs over all lattices of finite index, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function and s is a complex variable.

One way to prove (2.1) is to enumerate matrices in Hermite normal form; see [4, Section 1]. Its simplicity notwithstanding, this approach has two drawbacks: it is basis-dependent and is oblivious of an important set of intrinsic invariants, namely the *elementary divisors* of Λ with respect to the ambient lattice $\mathbb{Z}^n$.

Enumeration of lattices by their elementary divisors is achieved through suitable specializations of Igusa functions. The ***Igusa function of degree*** n is the rational function in variables $Z_1, \ldots, Z_n$

$$\mathsf{I}_n(Y; Z_1, \ldots, Z_n) = \sum_{I \subseteq [n]} \binom{n}{I}_Y \prod_{i \in I} \frac{Z_i}{1 - Z_i} \in \mathbb{Z}[Y](Z_1, \ldots, Z_n).$$

Here, $\binom{n}{I}_Y \in \mathbb{Z}[Y]$ is the Y-multinomial coefficient. The zeta function in (2.1) satisfies the following Euler product decomposition (see [26, Example 2.20]):

$$\zeta_{\mathbb{Z}^n}(s) = \prod_{p \text{ prime}} \mathsf{I}_n\left(p^{-1}; \left(p^{i(n-i-s)}\right)_{i \in [n]}\right).$$

Hall–Littlewood–Schubert series may be seen as substantial generalizations of Igusa functions. Indeed, one of their principal applications is to the enumeration of lattices $\Lambda \leqslant \mathbb{Z}^n$ by the elementary divisors of their intersections with all the members of a fixed complete isolated flag of $\mathbb{Z}^n$. As in the case of $\zeta_{\mathbb{Z}^n}(s)$, it suffices to solve this problem locally for all primes p, or equivalently for lattices in $\mathbb{Z}_p^n$, where $\mathbb{Z}_p$ is the ring of p-adic integers. More generally, we consider lattices over a compact discrete valuation ring (cDVR) $\mathfrak{o}$ of arbitrary characteristic.

In this local setup, the relevant elementary divisors are encoded by n partitions, one for each intersection. More precisely, let $V^\bullet$ denote the flag $0 = V^{(0)} < V^{(1)} < \cdots < V^{(n-1)} < V^{(n)} = \mathfrak{o}^n$ of submodules of V with $V^{(i+1)}/V^{(i)}$ torsion free and $V^{(i)}$ free of rank i. For a lattice $\Lambda \leqslant \mathfrak{o}^n$, denote the type of $\Lambda \cap V^{(i)}$ in $V^{(i)}$ by the partition $\lambda^{(i)}(\Lambda)$ of at most i parts. Given a partition $\lambda = (\lambda_j)$, we denote by $\mathrm{inc}(\lambda) = (\lambda_j - \lambda_{j+1})_j$ the composition comprising the increments of the parts of λ. Set $\mathrm{inc}(\lambda^\bullet(\Lambda)) = \left(\mathrm{inc}(\lambda^{(i)}(\Lambda))\right)_{i \in [n]} \in \mathbb{N}_0^{\binom{n+1}{2}}$. We introduce $\binom{n+1}{2}$ variables $\boldsymbol{Z} = (Z_{ij})_{1 \leqslant j \leqslant i \leqslant n}$ and set $\boldsymbol{Z}^{\mathrm{inc}(\lambda^\bullet(\Lambda))} = \prod_{i=1}^n \boldsymbol{Z}_i^{\mathrm{inc}(\lambda^{(i)}(\Lambda))}$.

Definition 2.1. The ***affine Schubert series of intersection type*** is

$$\mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}(\boldsymbol{Z}) = \sum_{\Lambda \leqslant \mathfrak{o}^n} \boldsymbol{Z}^{\mathrm{inc}(\lambda^\bullet(\Lambda))} \in \mathbb{Z}[\![\boldsymbol{Z}]\!], \tag{2.2}$$

where the sum runs over all finite-index sublattices Λ of $\mathfrak{o}^n$.

Remark 2.2. The term *affine Schubert series* is a nod to the fact that the defining sum (2.2) may (up to a factor) be interpreted as the generating function of a weight function on the vertices of the affine Bruhat–Tits building associated with the group $\mathrm{SL}_n(K)$, where K is the field of fractions of the cDVR $\mathfrak{o}$. Indeed, homothety classes of lattices in K^n form a natural model for the vertex set of the simplicial complex underlying this building. For an early exploitation of this perspective in the enumeration of lattices; see [24]. To what extent affine Schubert series are invariants of affine Schubert varieties remains an interesting open question.

Theorem B shows that the affine Schubert series $\mathsf{affS}^{\mathrm{in}}_{n,\mathfrak{o}}$ is a specialization of the Hall–Littlewood–Schubert series HLS_n. Given $C \subseteq [n]$, we set

$$d_n(C) = \left(\sum_{i\in[n]\setminus C} i\right) - \binom{n-\#C+1}{2}.$$

This is the dimension of the Schubert variety associated with C; see [10, page 1071]. We denote by $C(k)$ the kth smallest member of C. Set $C(\#C+1) = n+1$ and

$$\boldsymbol{Z}_{n,C} = \prod_{k=1}^{\#C} \prod_{\varepsilon=0}^{C(k+1)-C(k)-1} Z_{(C(k)+\varepsilon)k}.$$

Note that the (total) degree of $\boldsymbol{Z}_{n,C}$ is $n+1-C(1)$.

Theorem B. *For all cDVR $\mathfrak{o}$ with residue field cardinality q we have*

$$\mathsf{affS}^{\mathrm{in}}_{n,\mathfrak{o}}(\boldsymbol{Z}) = \mathsf{HLS}_n\left(q^{-1}, \left(q^{d_n(C)}\boldsymbol{Z}_{n,C}\right)_C\right).$$

Dually, we define $\mathsf{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}$ by recording the elementary divisor types of the projections onto a flag of reference. The duality between the two affine Schubert series is explained by a *jigsaw operation* on partitions and extended to tableaux, which is used to prove the following theorem.

Theorem C. *For all cDVR $\mathfrak{o}$ with residue field cardinality q we have*

$$\mathsf{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}(\boldsymbol{Z}) = \mathsf{HLS}_n\left(q^{-1}, \left(q^{d_n([n]\setminus C)}\boldsymbol{Z}_{n,C}\right)_C\right).$$

In particular, both $\mathsf{affS}^{\mathrm{in}}_{n,\mathfrak{o}}(\boldsymbol{Z})$ are $\mathsf{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}(\boldsymbol{Z})$ are rational functions in $\boldsymbol{Z}$ whose coefficients are polynomials in q. Key to the proof of Theorems B and C is to enumerate lattices Λ in $\mathfrak{o}^n$ by associated *intersection tableaux* resp. *projection tableaux*. They encode the information stored by the partitions $\mathrm{inc}(\lambda^{(i)}(\Lambda))$ for $i \in [n]$, in the intersection case and, analogously, by a vector of partitions encoding the types of the relevant projections. For the enumeration of the lattices with associated tableau T we deploy the leg polynomial $\Phi_T(Y)$ from Definition 1.1.

Combining Theorem A with Theorems B and C yields that the affine Schubert series also satisfies the following self-reciprocity property:

Corollary 2.3. *We have*

$$\left.\mathsf{affS}^{\mathrm{in}}_{n,\mathfrak{o}}(\boldsymbol{Z}^{-1})\right|_{q\to q^{-1}} = (-1)^n q^{\binom{n}{2}} \left(\prod_{i=1}^{n} Z_{ii}\right) \cdot \mathsf{affS}^{\mathrm{in}}_{n,\mathfrak{o}}(\boldsymbol{Z}),$$

$$\left.\mathsf{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}(\boldsymbol{Z}^{-1})\right|_{q\to q^{-1}} = (-1)^n q^{\binom{n}{2}} \left(\prod_{i=1}^{n} Z_{ii}\right) \cdot \mathsf{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}(\boldsymbol{Z}).$$

3 Hermite–Smith series

A further substitution of HLS_n pertains to the generating series enumerating lattices in $\mathfrak{o}^n$ according to their elementary divisor types and Hermite composition simultaneously. For the former, let $\lambda(\Lambda)$ be the partition encoding the elementary divisor type of a lattice $\Lambda \leqslant \mathfrak{o}^n$. For the latter, recall that Λ may be represented by a matrix $M \in \mathrm{Mat}_n(\mathfrak{o})$, whose rows record coordinates of generators of Λ with respect to some ordered $\mathfrak{o}$-basis of $\mathfrak{o}^n$. The coset $\mathrm{GL}_n(\mathfrak{o})M$ comprises all such matrices. Let

$$\delta(\Lambda) = (\delta_1(\Lambda), \dots, \delta_n(\Lambda)) \in \mathbb{N}_0^n$$

be the vector of valuations of the diagonal entries of any upper-triangular matrix in $\mathrm{GL}_n(\mathfrak{o})M$. The vector $\delta(\Lambda)$ is in fact an invariant of Λ and the flag $V^\bullet$ whose ith member is generated by the first i elements of the ordered basis. We thus call $\delta(\Lambda)$ the ***Hermite composition*** of Λ relative to $V^\bullet$.

Definition 3.1. For variables $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$ the ***Hermite–Smith series*** is

$$\mathrm{HS}_{n,\mathfrak{o}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\Lambda \leqslant \mathfrak{o}^n} \boldsymbol{x}^{\mathrm{inc}(\lambda(\Lambda))} \boldsymbol{y}^{\delta(\Lambda)} \in \mathbb{Z}[\![\boldsymbol{x}, \boldsymbol{y}]\!].$$

The Hermite–Smith series was first defined in [1, Section 1.3] because of its connection to the symplectic Hecke series; see Section 4. For $S \subseteq [n]$, let $\boldsymbol{x}_C = \prod_{i \in C} x_i$ and $\boldsymbol{y}_C = \prod_{i \in C} y_i$. Hermite–Smith series are instantiations of Hall–Littlewood–Schubert series:

Theorem D. *For $C \subseteq [n]$, set $C^* = \{n - i + 1 \mid i \in C\}$. We have*

$$\mathrm{HS}_{n,\mathfrak{o}}(\boldsymbol{x}, \boldsymbol{y}) = \mathsf{HLS}_n \left(q^{-1}, \left(q^{d_n(C)} x_{\# C} \boldsymbol{y}_{C^*} \right)_C \right).$$

4 Symplectic Hecke series

The Hecke series $\tau(Z)$ and its Fourier transforms $\hat{\tau}(\boldsymbol{s}, Z)$ associated with the groups of symplectic similitudes $\mathrm{GSp}_{2n}(F)$ over a local field F are the focus of [13, Section V.5], where Z and $\boldsymbol{s} = (s_0, \dots, s_n)$ are variables. In [13, V.5 (5.3)] Macdonald gives a formula for $\hat{\tau}(\boldsymbol{s}, Z)$ as a sum of 2^n rational functions in Z and $q^{-s_0}, \dots, q^{-s_n}$, where q is the residue field cardinality of the ring of integers $\mathfrak{o}$ of K. For a variable X, Macdonald exhibits a function

$$\mathsf{H}_{n,\mathfrak{o}}(\boldsymbol{x}, X) = \frac{\mathsf{H}_n^{\mathrm{num}}(q^{-1}, \boldsymbol{x}, X)}{\prod_{I \subseteq [n]} (1 - \boldsymbol{x}_I X)} \in \mathbb{Q}(\boldsymbol{x}, X),$$

where $\mathsf{H}_n^{\mathrm{num}}(Y, \boldsymbol{x}, X)$ is a polynomial of degree $2^n - 2$ in X, that satisfies

$$\hat{\tau}(s_0, \dots, s_n, Z) = \mathsf{H}_{n,\mathfrak{o}}(q^{-s_1}, \dots, q^{-s_n}, q^{N - s_0} Z) \tag{4.1}$$

for $N = \frac{1}{4}n(n+1)$. We extend the terminology (*symplectic*) *Hecke series* to the rational functions $\mathsf{H}_{n,\mathfrak{o}}$. We show that they are substitutions of HLS_n.

Theorem E. *For all cDVR $\mathfrak{o}$ with residue field cardinality q we have*

$$\mathsf{H}_{n,\mathfrak{o}}(\boldsymbol{x}, X)(1-X) = \mathsf{HLS}_n\left(q^{-1}, (\boldsymbol{x}_C X)_C\right).$$

In particular,

$$\mathsf{H}_n^{\mathrm{num}}(Y, \boldsymbol{x}, X) = \sum_{T \in \mathrm{rSSYT}_n} \Phi_T(Y) \prod_{C \in T} \boldsymbol{x}_C X \prod_{\varnothing \neq I \notin T} (1 - \boldsymbol{x}_I X) \in \mathbb{Z}[Y, \boldsymbol{x}, X].$$

In addition to providing an alternative to Macdonald's expression, this formula explicates a numerator of the rational function $\hat{\tau}(\boldsymbol{s}, X)$. It also reveals additional properties of the $\mathsf{H}_{n,\mathfrak{o}}$. Furthermore, Theorems A and E imply that the Hecke series also satisfies a self-reciprocity property.

Corollary 4.1. *For all cDVR $\mathfrak{o}$ with residue cardinality q we have*

$$\mathsf{H}_{n,\mathfrak{o}}\left(\boldsymbol{x}^{-1}, X^{-1}\right)\Big|_{q \to q^{-1}} = (-1)^{n+1} q^{\binom{n}{2}} x_1 \cdots x_n X^2 \cdot \mathsf{H}_{n,\mathfrak{o}}(\boldsymbol{x}, X).$$

One can show that the numerator polynomials $\mathsf{N}_n(Y, \boldsymbol{X})$ in (1.1) have no linear term in $\boldsymbol{X}$, that is, the coefficient, as an element of $\mathbb{Z}[Y]$, of X_I is 0 for all non-empty $I \subseteq [n]$. By Theorem E and Corollary 4.1, the coefficients of X and X^{2^n-3} in $\mathsf{H}_n^{\mathrm{num}}$ are both 0, thereby proving a conjecture of Panchishkin and Vankov concerning the Hecke series $\tau(Z)$ [16, Remark 1.3]. Moreover Corollary 4.1 proves a conjecture of Vankov [22, Remark 4] concerning the palindromicity of $\mathsf{H}_n^{\mathrm{num}}$.

In [14] we explore different interpretations of Hall–Littlewood–Schubert series as $\mathfrak{p}$-adic integrals. We show, specifically, that classical integrals over the integral $\mathfrak{p}$-adic points of groups of symplectic similitudes are instances of Hall–Littlewood–Schubert series. This yields a simplified proof of a combinatorial identity for these integrals in terms of Igusa functions, previously proven in [3]. We use a different expression of HLS_n as a $\mathfrak{p}$-adic integral to prove Theorem A.

5 Submodule and quiver representation zeta functions

Submodule zeta functions generalize the zeta function $\zeta_{\mathbb{Z}^n}(s)$ introduced in Section 2. Whereas the latter enumerates all sublattices in $\mathbb{Z}^n$, the former enumerate submodules of finite index that are invariant under an integral matrix algebra.

Before we set out our contributions to this class of zeta functions, we briefly sample a few of the milestones in the development of this class of Dirichlet series. A classical prototype is Dedekind's zeta function associated to a number field, enumerating ideals in the number field's ring of integers. Solomon was interested in submodule zeta functions in the context of integral representation theory; see [21]. Grunewald, Segal, and Smith studied global and local submodule zeta functions associated with nilpotent Lie rings in [9], pioneering tools from model theory and $\mathfrak{p}$-adic integration. An algebro-geometric approach was taken in du Sautoy and Grunewald's seminal paper [5]. Rossmann turned a toroidal vantage point into theoretical [17] and practical [18] advances. The second author studied submodule zeta functions associated with nilpotent matrix algebras of class 2 via affine Bruhat–Tits buildings [24] and established self-reciprocity results akin to Theorem A in [25]. Both authors introduced zeta functions as invariants of hyperplane arrangements in [15] and established connections with zeta functions associated with hypergraphs in [20].

Algebraic geometry, notably $\mathfrak{p}$-adic integration, has been the prevalent source of methodology in the development of the theory of submodule zeta functions in the recent decade [19]. We argue that Hall–Littlewood–Schubert series HLS_n are a powerful new tool in the study of these and related Dirichlet generating series. We substantiate this with Theorem F, which we believe to be one of many instances of this phenomenon.

As explained in [11, Section 1.3.3], submodule zeta functions are exactly the zeta functions of integral quiver representations. To explain the latter, recall that a ***quiver*** Q is a finite directed graph with vertex set Q_0 and arrow set Q_1. For $\alpha \in Q_1$, write $h(\alpha) \in Q_0$ and $t(\alpha)$ for the respective head and tail of α: if $\alpha : i \to j$, then $h(\alpha) = i$ and $t(\alpha) = j$. Let R be a commutative ring. An ***R-representation*** of Q is a collection $U = (U_\iota)_{\iota \in Q_0}$ of R-modules U_i, together with an R-module homomorphisms $f_\alpha : U_{t(\alpha)} \to U_{h(\alpha)}$ for each $\alpha \in Q_1$. An R-representation U', with modules U_i' and homomorphisms f_α', is a ***subrepresentation*** of U if $U_j' \leqslant U_j$ with inclusion $\iota_j : U_j' \hookrightarrow U_j$ for all $j \in Q_0$ and $f_\alpha \iota_j = \iota_k f_\alpha'$ for all arrows $\alpha : j \to k$. In this case, we write $U' \leqslant U$. The ***index*** of U' in U is the product of the indices $|U_i : U_i'|$ for each $i \in Q_0$.

The *representation zeta function* $\zeta_U(\boldsymbol{s})$ associated with a fixed R-representation U of a quiver Q was first introduced in [11] for the case when R is a global or local ring of integers and the U_i free, finite-rank R-modules; see [11, (1.1)]. Let $\boldsymbol{s} = (s_i)_{i \in Q_0}$ be complex variables. The representation zeta function is defined as

$$\zeta_U(\boldsymbol{s}) = \sum_{U' \leqslant U} \prod_{i \in Q_0} |U_i : U_i'|^{-s_i}, \tag{5.1}$$

where the sum runs over finite index subrepresentations of U. Certain substitutions of the rational functions HLS_n yield concrete formulae for the (local) representation zeta functions of various quiver representations. We exemplify this with certain representations of dual star quivers.

For $n \in \mathbb{N}$, the *dual star quiver* S_n^* is the quiver with vertex set $[n]$ and arrows $\alpha_i : i \to n$ for all $i \in [n-1]$. We define a representation $V_n(\mathfrak{o})$ of S_n^*, as follows: let $V_i = \mathfrak{o}^i$ for all

$i \in [n]$, and let $f_{\alpha_i} : \mathfrak{o}^i \to \mathfrak{o}^n$ be an embedding whose images form a complete isolated flag in $V_n = \mathfrak{o}^n$.

Theorem F. *For $C \subseteq [n]$, set $C_0 = C \cup \{0\}$ and let $v_C = (\max(C_0 \cap [i]_0))_{i=1}^n \in \mathbb{N}_0^n$. For the $\mathfrak{o}$-representation $V_n(\mathfrak{o})$ of S_n^* as above, we have*

$$\zeta_{V_n(\mathfrak{o})}(\boldsymbol{s}) = \mathsf{HLS}_n\left(q^{-1}, \left(q^{d_n(C) - v_C \cdot \boldsymbol{s}}\right)_C\right) \prod_{i=1}^{n-1} \zeta_{\mathfrak{o}^i}(s_i).$$

6 Tableaux and Bruhat orders

Hall–Littlewood–Schubert series are defined as finite sums over reduced tableaux. Identifying this index set with the set of chains in a poset opens further combinatorial and topological vantage points.

We define a partial order $\sqsubseteq$ which we call *tableaux order* on the set T_n of non-empty subsets of $[n]$ and explore the topological properties of its associated order complex. In this poset structure, we compare non-empty subsets A and B of $[n]$, written $A \sqsubseteq B$, if A and B arise as labels of adjacent columns in some tableau $T \in \mathrm{SSYT}_n$. This refines the usual containment relation $\supseteq$ on $[n]$: if $A \supseteq B$, then $A \sqsubseteq B$. This partial order has been studied in different contexts and is also known as the *Gale order*; see [8].

Hall–Littlewood–Schubert series are Y-analogs of Stanley–Reisner rings of a simplicial complex, namely the order complex $\Delta(\mathsf{T}_n)$ of the poset T_n. As an abstract simplicial complex, $\Delta(\mathsf{T}_n)$ is isomorphic to the set of rSSYT_n of reduced tableaux with labels in $[n]$. We denote by $|\Delta(\mathsf{T}_n)|$ a geometric realization of $\Delta(\mathsf{T}_n)$.

Theorem 6.1. *The simplicial complex $|\Delta(\mathsf{T}_n)|$ is Cohen–Macaulay over $\mathbb{Z}$ and homeomorphic to an $\left(\binom{n+1}{2} - 1\right)$-ball. The number of maximal flags in $\Delta(\mathsf{T}_n)$ is*

$$\frac{\binom{n+1}{2}! \cdot \prod_{a=1}^{n-1}(a!)}{\prod_{b=1}^{n}((2b-1)!)}.$$

At the heart of the proof of Theorem 6.1 is a poset isomorphism between T_n and a poset arising from the parabolic quotient of the hyperoctahedral group of degree n by its maximal symmetric group. The partial order on that set is given by the Bruhat order.

In [14] we study special values of Hall–Littlewood–Schubert series. We focus on univariate series obtained by setting all the X_C to X and Y to one of 0, 1, or -1. In the case $Y = 0$ the Cohen–Macaulay property of certain (Stanley–Reisner) rings implies the non-negativity of the relevant series' numerators. In the other cases $Y = 1$ or $Y = -1$, we formulate non-negativity conjectures that seem to transcend the remit of Stanley–Reisner rings of simplicial complexes.

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