

FLAG HILBERT–POINCARÉ SERIES AND IGUSA ZETA FUNCTIONS OF HYPERPLANE ARRANGEMENTS

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ABSTRACT. We introduce and study a class of multivariate rational functions associated with hyperplane arrangements, called flag Hilbert–Poincaré series. These series are intimately connected with Igusa local zeta functions of products of linear polynomials, and their motivic and topological relatives. Our main results include a self-reciprocity result for central arrangements defined over fields of characteristic zero. We also prove combinatorial formulae for a specialization of the flag Hilbert–Poincaré series for irreducible Coxeter arrangements of types A, B, and D in terms of total partitions of the respective types. We show that a different specialization of the flag Hilbert–Poincaré series, which we call the coarse flag Hilbert–Poincaré series, exhibits intriguing nonnegativity features and—in the case of Coxeter arrangements—connections with Eulerian polynomials. For numerous classes and examples of hyperplane arrangements, we determine their (coarse) flag Hilbert–Poincaré series. Some computations were aided by a SageMath package we developed.

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1. INTRODUCTION

A hyperplane arrangement over a field \mathbb{K} is a finite set \mathcal{A} of affine hyperplanes in \mathbb{K}^d for some integer $d =: \dim(\mathcal{A})$. In this paper we introduce and study a multivariate rational function $\text{fHP}_{\mathcal{A}}(Y, \mathbf{T}) \in \mathbb{Q}(\mathbf{T})[Y]$, called the *flag Hilbert–Poincaré series* of \mathcal{A} , encompassing much of the topology and combinatorics of \mathcal{A} .

In order to define $\text{fHP}_{\mathcal{A}}$, we introduce some further notation. Let $\mathcal{L}(\mathcal{A})$ be the intersection poset of \mathcal{A} , ordered by reverse-inclusion. Two hyperplane arrangements are *equivalent* if their intersection posets are isomorphic. We denote by $\hat{0}$ (resp. $\hat{1}$) the bottom (resp. top) element of a poset (provided $\hat{1}$ exists). Observe that $\hat{1} \in \mathcal{L}(\mathcal{A})$ if and only if \mathcal{A} is central, i.e. $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$. Define $\tilde{\mathcal{L}}(\mathcal{A}) = \mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}$

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and $\bar{\mathcal{L}}(\mathcal{A}) = \mathcal{L}(\mathcal{A}) \setminus \{\hat{0}, \hat{1}\}$. For $x \in \mathcal{L}(\mathcal{A})$, we write $\text{rk}(x) := \text{rk}_{\mathcal{L}(\mathcal{A})}(x)$ for the *rank* of x , viz. the supremum over the lengths of all chains from $\hat{0}$ to x . For a poset P , the *order complex* $\Delta(P)$ associated with P is the simplicial complex with vertex set P , whose simplices are the flags of P . For $x \in \mathcal{L}(\mathcal{A})$, define hyperplane arrangements

$$\begin{aligned} \mathcal{A}_x &= \{H \in \mathcal{A} \mid x \subseteq H\}, & (\text{subarrangement}) \\ \mathcal{A}^x &= \{x \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_x, x \cap H \neq \emptyset\} & (\text{restriction}). \end{aligned}$$

Set $\mathcal{A}_{\emptyset} := \mathcal{A}$. Interlacing these constructions we obtain, for $x, y \in \mathcal{L}(\mathcal{A})$, the arrangement $\mathcal{A}_{xy}^x := (\mathcal{A}^x)_y = (\mathcal{A}_y)^x$. Recall further the *Poincaré polynomial*

$$(1.1) \quad \pi_{\mathcal{A}}(Y) = \sum_{x \in \mathcal{L}(\mathcal{A})} \mu(x) (-Y)^{\text{rk}(x)} \in \mathbb{Z}[Y]$$

associated with \mathcal{A} , where μ is the Möbius function on $\mathcal{L}(\mathcal{A})$; cf. [19, Def. 2.48]. The Poincaré polynomial is closely related to the characteristic polynomial $\chi_{\mathcal{A}}(Y)$ of \mathcal{A} via the identity (see [19, Def. 2.52])

$$(1.2) \quad \chi_{\mathcal{A}}(Y) = Y^d \pi_{\mathcal{A}}(-Y^{-1}).$$

We require the following flag generalization: for $F = (x_1 < \dots < x_\ell) \in \Delta(\mathcal{L}(\mathcal{A}))$ (possibly empty), set $x_0 = \hat{0}$ and $x_{\ell+1} = \emptyset$, and define

$$(1.3) \quad \pi_F(Y) = \prod_{k=0}^{\ell} \pi_{\mathcal{A}_{x_{k+1}}^{x_k}}(Y) \in \mathbb{Z}[Y].$$

The following function is the main protagonist of the current paper.

Definition 1.1. Let $\mathbf{T} := (T_x)_{x \in \tilde{\mathcal{L}}(\mathcal{A})}$ be indeterminates. The *flag Hilbert–Poincaré series* associated with \mathcal{A} is

$$\text{fHP}_{\mathcal{A}}(Y, \mathbf{T}) = \sum_{F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))} \pi_F(Y) \prod_{x \in F} \frac{T_x}{1 - T_x} \in \mathbb{Q}(\mathbf{T})[Y].$$

If \mathcal{A} is central, then

$$\text{fHP}_{\mathcal{A}}(Y, \mathbf{T}) = \frac{1}{1 - T_{\hat{1}}} \sum_{F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))} \pi_F(Y) \prod_{x \in F} \frac{T_x}{1 - T_x}.$$

We remark that $\text{fHP}_{\mathcal{A}}(0, \mathbf{T})$ is the (fine) Hilbert series of the Stanley–Reisner ring of the order complex of $\tilde{\mathcal{L}}(\mathcal{A})$; see Proposition 1.9.

The following self-reciprocity result for central arrangements over fields of characteristic zero is our first main theorem. The *rank* of \mathcal{A} , denoted by $\text{rk}(\mathcal{A})$, is the rank of a maximal element of $\mathcal{L}(\mathcal{A})$.

Theorem A (Self-reciprocity). *Let \mathcal{A} be a central hyperplane arrangement over a field of characteristic zero. Then*

$$(1.4) \quad \text{fHP}_{\mathcal{A}}\left(Y^{-1}, (T_x^{-1})_{x \in \tilde{\mathcal{L}}(\mathcal{A})}\right) = (-Y)^{-\text{rk}(\mathcal{A})} T_{\hat{1}} \cdot \text{fHP}_{\mathcal{A}}(Y, \mathbf{T}).$$

Remark 1.2. The restriction to fields of characteristic zero reflects our method of proof (see Sections 2 and 3) rather than any known counterexamples in positive characteristic. Indeed, the Fano arrangement, comprising the seven points in the projective plane over \mathbb{F}_2 , has no equivalent arrangement over characteristic zero, yet satisfies (1.4); see Section 4.7.

Simple examples show that the kind of *self*-reciprocity expressed in Theorem A is not to be expected for noncentral arrangements; see Section 4.1. It remains of interest to investigate the possibilities of reciprocity results linking $\text{fHP}_{\mathcal{A}}(Y^{-1}, \mathbf{T}^{-1})$ to the flag Hilbert–Poincaré series of other (“reciprocal”) hyperplane arrangements. One may also want to explore self-reciprocity phenomena for (central) arrangements

over fields of positive characteristic and potential connections to reciprocity phenomena for other rational generating functions such as the ones introduced in [1, Sec. 5].

Various substitutions of the variables of the flag Hilbert–Poincaré series yield connections to seemingly different enumeration problems: first, we explain in Section 1.1 that flag Hilbert–Poincaré series encode the same information as certain \mathfrak{p} -adic integrals associated with hyperplane arrangements (see Theorem B). In Section 1.2 we explicate the specific connections to the well-studied class of (both uni- and multivariate) *Igusa local zeta functions* associated with products of linear polynomials, and their cousins, the *topological zeta functions* (see Corollary 1.5).

Second, we discuss in Section 1.3 an alternative combinatorial formula (see Theorem C) for specific multivariate substitutions, viz. *atom zeta functions*, associated with *classical* Coxeter arrangements—viz. irreducible Coxeter arrangements of types A, B, or D—in terms of total partitions and rooted trees.

Third, we focus in Section 1.4 on *coarse flag Hilbert–Poincaré series*, viz. the bivariate “coarsening” of the flag Hilbert–Poincaré series $\text{fHP}_{\mathcal{A}}(Y, \mathbf{T})$ obtained by setting $T_x = T$ for all x . Our Theorem D presents coarse flag Hilbert–Poincaré series associated with Coxeter arrangements as “ Y -analogs” of Hilbert series of the Stanley–Reisner rings of the first barycentric subdivisions of standard simplices. On the level of rational generating functions, this is reflected by an intriguing connection with Eulerian polynomials.

Remark 1.3. Another bivariate substitution relates the flag Hilbert–Poincaré series of a hyperplane arrangement \mathcal{A} with the motivic zeta function $Z_{M(\mathcal{A})}(Y, T)$ introduced in [16, Def. 1.1] associated with the (representable) matroid $M(\mathcal{A})$ determined by \mathcal{A} . Indeed, we have

$$\text{fHP}_{\mathcal{A}} \left(-Y^{-1}, \left(Y^{-\text{rk}(x)} T^{|\mathcal{A}_x|} \right)_{x \in \tilde{\mathcal{L}}(\mathcal{A})} \right) = Z_{M(\mathcal{A})}(Y, T).$$

Our Theorem A implies [16, Thm. 1.6] in this case. See also Remark 3.4.

1.1. Flag Hilbert–Poincaré series and \mathfrak{p} -adic integrals. For general arrangements \mathcal{A} over fields of characteristic zero, the functions $\text{fHP}_{\mathcal{A}}$ are universal objects from which various \mathfrak{p} -adic integrals associated with \mathcal{A} may be obtained via specializations. To discuss this connection, we first recall some representability properties of hyperplane arrangements.

If K is a field and \mathcal{A}_K is a hyperplane arrangement defined over K such that $\mathcal{L}(\mathcal{A}) \cong \mathcal{L}(\mathcal{A}_K)$ as posets, then we say that \mathcal{A} is K -representable and call \mathcal{A}_K a K -representation of \mathcal{A} . If, as we now assume, \mathcal{A} is a hyperplane arrangement defined over a field \mathbb{K} of characteristic zero, there exists a finite extension K of \mathbb{Q} such that \mathcal{A} is K -representable; cf. [20, Prop. 6.8.11]. Having fixed such a representation of \mathcal{A} , we may further assume, without loss of generality, that each $H \in \mathcal{A}$ is of the form $H = V(L)$, where $L(\mathbf{X}) = c_L + \sum_{j=1}^d \alpha_{L,j} X_j \in \mathcal{O}_K[\mathbf{X}]$ is an affine linear polynomial over \mathcal{O}_K , the ring of integers of the number field K . These choices allow us, in fact, to identify the arrangement \mathcal{A} with the collection of polynomials L arising in this way. We will use this freedom frequently.

In the sequel we denote by \mathfrak{o} a compact discrete valuation ring (cDVR) with an \mathcal{O}_K -module structure. This could be a finite extension of the completion $\mathcal{O}_{K,\mathfrak{p}}$ of \mathcal{O}_K at a nonzero prime ideal \mathfrak{p} (in characteristic zero) or a power series ring of the form $\mathbb{F}_q[[X]]$, where \mathbb{F}_q is the residue field of such a ring (in positive characteristic).

Denoting by \mathfrak{p} the unique maximal ideal of \mathfrak{o} , we write $\mathcal{A}(\mathfrak{o}/\mathfrak{p})$ for the reduction of \mathcal{A} modulo \mathfrak{p} . If $\mathcal{L}(\mathcal{A}) \cong \mathcal{L}(\mathcal{A}(\mathfrak{o}/\mathfrak{p}))$, then \mathcal{A} is said to have *good reduction over \mathbb{F}_q* , provided $\mathfrak{o}/\mathfrak{p}$ has cardinality q . It is well-known that \mathcal{A} has good reduction over \mathbb{F}_q for all such q not divisible by finitely many (“bad”) primes; cf. [28, Chap. 5.1].

We now explain the connection between the flag Hilbert–Poincaré series $\text{fHP}_{\mathcal{A}}$ and various (multi- and univariate) \mathfrak{p} -adic integrals associated with the hyperplane arrangement \mathcal{A} .

Definition 1.4. The *analytic zeta function* of \mathcal{A} over \mathfrak{o} is

$$\zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s}) = \int_{\mathfrak{o}^{\dim(\mathcal{A})}} \prod_{x \in \tilde{\mathcal{L}}(\mathcal{A})} \|\mathcal{A}_x\|^{s_x} |\mathrm{d}\mathbf{X}|,$$

where s_x is a complex variable for each $x \in \tilde{\mathcal{L}}(\mathcal{A})$, further $\|\mathcal{X}\| := \max\{|f| \mid f \in \mathcal{X}\}$ for a finite set $\mathcal{X} \subset \mathfrak{o}$, and $|\mathrm{d}\mathbf{X}|$ is the additive Haar measure on $\mathfrak{o}^{\dim(\mathcal{A})}$, normalized so that $\mathfrak{o}^{\dim(\mathcal{A})}$ has measure 1.

Our next main result establishes that the functions $\zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s})$ and $\text{fHP}_{\mathcal{A}}(Y, \mathbf{T})$ determine each other, in the following precise sense.

Theorem B. *Let \mathcal{A} be a hyperplane arrangement over a number field K . For indeterminates $\mathbf{s} := (s_x)_{x \in \tilde{\mathcal{L}}(\mathcal{A})}$ and $\mathbf{r} := (r_x)_{x \in \tilde{\mathcal{L}}(\mathcal{A})}$ and $x \in \tilde{\mathcal{L}}(\mathcal{A})$, let*

$$(1.5) \quad g_x(\mathbf{s}) = \text{rk}(x) + \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y,$$

$$(1.6) \quad h_x(\mathbf{r}) = \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} (r_y - \text{rk}(y)) \mu(y, x).$$

If \mathfrak{o} is a cDVR and an \mathcal{O}_K -module with residue field cardinality q such that \mathcal{A} has good reduction over \mathbb{F}_q , then

$$(1.7) \quad \zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s}) = \text{fHP}_{\mathcal{A}} \left(-q^{-1}, \left(q^{-g_x(\mathbf{s})} \right)_{x \in \tilde{\mathcal{L}}(\mathcal{A})} \right),$$

$$(1.8) \quad \text{fHP}_{\mathcal{A}} \left(-q^{-1}, \left(q^{-r_x} \right)_{x \in \tilde{\mathcal{L}}(\mathcal{A})} \right) = \zeta_{\mathcal{A}(\mathfrak{o})} \left((h_x(\mathbf{r}))_{x \in \tilde{\mathcal{L}}(\mathcal{A})} \right).$$

A consequence of Theorem B is that $\text{fHP}_{\mathcal{A}}(Y, \mathbf{T})$ provides an *explicit*, combinatorial formula for the multivariate rational function $\zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s})$. General formulae for \mathfrak{p} -adic integrals associated with polynomial mappings—not necessarily defined by linear forms—are typically obtained via resolutions of singularities (or log-principalizations) of the varieties these mappings define; see, for instance, [32] and compare Section 3.1. Our combinatorial approach obliterates the need for these, in general quite unwieldy, algebro-geometric tools and the choices they require.

We will prove Theorem B in Section 2. The interpretation of $\text{fHP}_{\mathcal{A}}$ in terms of the \mathfrak{p} -adic integrals $\zeta_{\mathcal{A}(\mathfrak{o})}$ expressed by Theorem B is key to our proof of Theorem A in Section 3.

We record numerous examples of the rational functions $\text{fHP}_{\mathcal{A}}$ in Section 4. The case of Boolean arrangements $\mathcal{A} = \mathbf{A}_1^n$ is of particular interest: specific substitutions of the functions $\text{fHP}_{\mathbf{A}_1^n}$ arise in the study [24] of the average sizes of kernels of generic matrices with support constraints over finite quotients of cDVRs; see Section 4.8.

1.2. Igusa and topological zeta functions. Assume now, as in Section 1.1, that \mathcal{A} is a K -representation and \mathfrak{o} is a cDVR and an \mathcal{O}_K -module. An important specialization of the multivariate function $\zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s})$ yields the (univariate) *Igusa local zeta function* (over \mathfrak{o}) associated with the product $f_{\mathcal{A}}(\mathbf{X}) := \prod_{L \in \mathcal{A}} L(\mathbf{X})$ of linear polynomials $L \in \mathcal{O}_K[\mathbf{X}]$ (see [10]):

$$(1.9) \quad Z_{f_{\mathcal{A}}, \mathfrak{o}}(s) := \zeta_{\mathcal{A}(\mathfrak{o})} \left((s \cdot \delta_{|A_x|=1})_{x \in \tilde{\mathcal{L}}(\mathcal{A})} \right) = \int_{\mathfrak{o}^{\dim(\mathcal{A})}} |f_{\mathcal{A}}|^s |\mathrm{d}\mathbf{X}|;$$

here s is a complex variable. Motivic zeta functions related with such integrals have been studied and can be used to understand the topological zeta function $Z_{f_{\mathcal{A}}}^{\text{top}}(s)$

associated with $f_{\mathcal{A}}(\mathbf{X})$. This is executed, for example, in [6] for generic central arrangements, among others; cf. Section 4.6.

As a consequence of Theorem B, we derive the (multivariate and univariate) topological zeta function of a hyperplane arrangement over a field of characteristic zero. From Theorem B, $\text{fHP}_{\mathcal{A}}(Y, \mathbf{T})$ defines a system of local zeta functions of Deneff type in the sense of Rossmann [22, Sec. 5]. Associated with \mathcal{A} is thus a unique rational function $\zeta_{\mathcal{A}}^{\text{top}}(\mathbf{s}) \in \mathbb{Q}(\mathbf{s})$, interpretable as the limit “ $q \rightarrow 1$ ” and called the *multivariate topological zeta function* of \mathcal{A} . Analogous to (1.9), the topological zeta function $Z_{f_{\mathcal{A}}}^{\text{top}}(s)$ is a specialization of $\zeta_{\mathcal{A}}^{\text{top}}(\mathbf{s})$. For further details, see Section 2.2.

For $F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))$, set

$$(1.10) \quad \pi_F^{\circ}(Y) := \frac{\pi_F(Y)}{(1+Y)^{|F|}}, \quad \bar{\pi}_F^{\circ}(Y) := \frac{\pi_F(Y)}{(1+Y)^{|F|+1}}.$$

In fact, $\pi_F^{\circ}(Y)$ is a polynomial in Y , and if \mathcal{A} is central and $F \in \Delta(\bar{\mathcal{L}}(\mathcal{A}))$, then $\bar{\pi}_F^{\circ}(Y)$ is also a polynomial in Y ; see Lemma 2.3.

Corollary 1.5. *Let \mathcal{A} be a hyperplane arrangement over a field of characteristic zero, and let $\mathbf{s} = ((s_x)_{x \in \tilde{\mathcal{L}}(\mathcal{A})})$ be indeterminates. Then the multivariate topological zeta function of \mathcal{A} is*

$$(1.11) \quad \zeta_{\mathcal{A}}^{\text{top}}(\mathbf{s}) = \sum_{F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))} \pi_F^{\circ}(-1) \prod_{x \in F} \frac{1}{\text{rk}(x) + \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y}.$$

If \mathcal{A} is also central, then

$$(1.12) \quad \zeta_{\mathcal{A}}^{\text{top}}(\mathbf{s}) = \frac{1}{\text{rk}(\mathcal{A}) + \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A})} s_y} \sum_{F \in \Delta(\bar{\mathcal{L}}(\mathcal{A}))} \bar{\pi}_F^{\circ}(-1) \prod_{x \in F} \frac{1}{\text{rk}(x) + \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y}.$$

We see in particular that if $f \in \mathbb{C}[\mathbf{X}]$ is the product of linear polynomials, then the topological zeta function $Z_f^{\text{top}}(s)$ depends only on the combinatorics of the associated (multi-) arrangement; see also [5, Prop. 2.2]. This is also along the lines and consistent with work of van der Veer, who proves that topological zeta functions for general matroids are independent of the underlying building set; see [31].

The zeta functions $Z_f(s)$ and $Z_f^{\text{top}}(s)$ lie at the heart of the Monodromy and Topological Monodromy Conjectures [15], connecting the singularities of $\{f = 0\}$ with the local monodromy action on the Milnor fibers of f and relating the roots of the Bernstein–Sato polynomial $b_f(s)$ with the (real parts of the) poles of $Z_f(s)$; see [10] for a general introduction. Budur, Mustață, and Teitler [4] proved that arbitrary complex hyperplane arrangements satisfy part of the Topological Monodromy Conjecture and further reduced the remaining part to the so-called Strong Monodromy Conjecture [4, Conjecture 1.2], which is concerned with the existence of a specific, combinatorially defined root of $b_f(s)$. The Strong Monodromy Conjecture is known to hold for some classes of hyperplane arrangements [5, Theorem 1.4]—notably for Weyl arrangements [2, Theorem 1.1]—but it is, in general, open. Wu recently proved a multivariate version of the Topological Monodromy Conjecture for hyperplane arrangements; cf. [36, Thm. 1.7].

1.3. Atom zeta functions. The specialization $Z_{f_{\mathcal{A}, \circ}}(s)$ defined in (1.9) loses sight of all variables not corresponding to atoms (i.e. minimal elements in $\tilde{\mathcal{L}}(\mathcal{A})$) and cannot distinguish atoms. Just as the multivariate topological zeta function (1.11) refines the univariate topological zeta function, we consider the following, slightly more distinguishing \mathfrak{p} -adic specialization of $\zeta_{\mathcal{A}(\circ)}(\mathbf{s})$.

Definition 1.6. The *atom zeta function* of \mathcal{A} is

$$\zeta_{\mathcal{A}(\mathfrak{o})}^{\text{at}}((s_L)_{L \in \mathcal{A}}) = \zeta_{\mathcal{A}(\mathfrak{o})} \left((s_x \cdot \delta_{|\mathcal{A}_x|=1})_{x \in \tilde{\mathcal{L}}(\mathcal{A})} \right) = \int_{\mathfrak{o}^{\dim(\mathcal{A})}} \prod_{L \in \mathcal{A}} |L|^{s_L} |\mathrm{d}\mathbf{X}|.$$

Here we identified atoms with elements $L \in \mathcal{A}$. We note that the independent variables $(s_L)_{L \in \mathcal{A}}$ allow for the treatment of multi-arrangements in the sense of [6].

We remark that the atom zeta function is the finest coarsening of the multivariate zeta function $\zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s})$ that is, in general, multiplicative with respect to direct products of hyperplane arrangements. Namely, if \mathcal{A} and \mathcal{A}' are arrangements of hyperplanes in (disjoint vector spaces) K^d and $K^{d'}$ and \mathfrak{o} is as above, then, by Fubini's theorem,

$$\zeta_{(\mathcal{A} \times \mathcal{A}')(\mathfrak{o})}^{\text{at}}((\mathbf{s}, \mathbf{s}')) = \zeta_{\mathcal{A}(\mathfrak{o})}^{\text{at}}(\mathbf{s}) \zeta_{\mathcal{A}'(\mathfrak{o})}^{\text{at}}(\mathbf{s}').$$

Our next result paraphrases an explicit combinatorial formula for atom zeta functions associated with classical Coxeter arrangements; see Section 5.1 for definitions and the precise version (Theorem 5.6). There we define, in particular, for $n \in \mathbb{N}$, the sets $\mathrm{TP}_{\mathbf{X},n}$ of total partitions of type \mathbf{X}_n ; for type $\mathbf{X} = \mathbf{A}$, these are also defined in [27, Example 5.2.5] and related to Schröder's fourth problem.

Theorem C. Let $\mathbf{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{D}\}$ and $n \in \mathbb{N}$, with $n \geq 2$ if $\mathbf{X} = \mathbf{D}$. Then there exist, for all $\tau \in \mathrm{TP}_{\mathbf{X},n}$, explicitly determined polynomials $\pi_{\mathbf{X},\tau}(Y) \in \mathbb{Z}[Y]$ and products of geometric progressions $\mathbf{Cgp}_{\mathbf{X},\tau}(Z, (T_L)_{L \in \mathbf{X}_n})$ such that the following holds: for all $c\mathrm{DVR}$ \mathfrak{o} with residue field cardinality q , assumed to be odd unless $\mathbf{X} = \mathbf{A}$,

$$\zeta_{\mathbf{X}_n(\mathfrak{o})}^{\text{at}}((s_L)_{L \in \mathbf{X}_n}) = \frac{1}{1 - q^{-n - \sum_{L \in \mathbf{X}_n} s_L}} \sum_{\tau \in \mathrm{TP}_{\mathbf{X},n}} \pi_{\mathbf{X},\tau}(-q^{-1}) \mathbf{Cgp}_{\mathbf{X},\tau}(q^{-1}, (q^{-s_L})_{L \in \mathbf{X}_n}).$$

One may use these formulae to obtain alternative combinatorial formulae for the (multi- and univariate) topological zeta functions given in Corollary 1.5. We leave the details to the reader and refer to Example 5.8.

As a consequence of Theorem C, we obtain in Corollary 5.7 an explicit formula for Igusa's local zeta function $Z_{f_{\mathbf{A}_n}, \mathfrak{o}}(s)$ in terms of unlabeled rooted trees with $n+1$ leaves. In Corollary 5.9 we express the atom zeta functions $\zeta_{\mathbf{B}_n(\mathfrak{o})}^{\text{at}}(\mathbf{s})$ and $\zeta_{\mathbf{D}_n(\mathfrak{o})}^{\text{at}}(\mathbf{s})$ as sums over $\mathrm{TP}_{\mathbf{A},n}$, a set notably smaller than $\mathrm{TP}_{\mathbf{B},n}$ and $\mathrm{TP}_{\mathbf{D},n}$.

1.4. Coarse flag Hilbert–Poincaré series. Consider now the bivariate specialization of the flag Hilbert–Poincaré series $\mathrm{fHP}_{\mathcal{A}}$ obtained by setting $T_x = T$ for each $x \in \tilde{\mathcal{L}}(\mathcal{A})$:

Definition 1.7. The *coarse flag Hilbert–Poincaré series* of \mathcal{A} is

$$\mathrm{cfHP}_{\mathcal{A}}(Y, T) = \sum_{F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))} \pi_F(Y) \left(\frac{T}{1-T} \right)^{|F|} \in \mathbb{Q}(T)[Y].$$

We define the polynomial $\mathcal{N}_{\mathcal{A}}(Y, T) \in \mathbb{Q}[Y, T]$ by the formula

$$(1.13) \quad \mathrm{cfHP}_{\mathcal{A}}(Y, T) = \frac{\mathcal{N}_{\mathcal{A}}(Y, T)}{(1-T)^{\mathrm{rk}(\mathcal{A})}}.$$

In Section 6 we explore a number of remarkable properties of these rational functions, including nonnegativity features of $\mathcal{N}_{\mathcal{A}}(Y, T)$ and—in the case of Coxeter arrangements—connections with Eulerian and Stirling numbers.

In Proposition 1.9, we observe that $\mathcal{N}_{\mathcal{A}}(0, T)$ has nonnegative coefficients. Its proof is based on the fact that $\mathrm{cfHP}_{\mathcal{A}}(0, T)$ is the coarse Hilbert series of the Stanley–Reisner ring of the order complex of $\tilde{\mathcal{L}}(\mathcal{A})$. The Cohen–Macaulayness of this complex implies the nonnegativity of the associated h -vector, i.e. the coefficients of $\mathcal{N}_{\mathcal{A}}(0, T)$.

Recall that the n th *Eulerian polynomial* $E_n(T)$ is defined via

$$(1.14) \quad E_n(T) = \sum_{w \in S_n} T^{\text{des}(w)} \in \mathbb{Z}[T],$$

where $\text{des}(w) := |\{i \in [n-1] \mid w(i) > w(i+1)\}|$. Let $S(n, k)$ be the Stirling number of the second kind; see [29, Sec. 1.9]. It is well-known that

$$(1.15) \quad \frac{E_n(T)}{(1-T)^n} = \sum_{k=1}^n k! S(n, k) \left(\frac{T}{1-T} \right)^{k-1}$$

is the (coarse) Hilbert series of the Stanley–Reisner ring $\mathbb{F}[\text{sd}(\partial\Delta_{n-1})]$ associated with the first barycentric subdivision of the boundary of the $(n-1)$ -dimensional simplex Δ_{n-1} over a field \mathbb{F} ; cf. [21, Thm. 9.1].

A real hyperplane arrangement \mathcal{A} is a *Coxeter arrangement* if the set of reflections across its hyperplanes fixes \mathcal{A} and forms a finite Coxeter group under composition. We call \mathcal{A} *irreducible* if it is not a direct product of two nontrivial arrangements. Finite Coxeter arrangements may be decomposed as direct products of irreducible Coxeter arrangements. The latter come in two classes: *classical* Coxeter arrangements of types A, B, or D and *exceptional* Coxeter arrangements of types E_6 , E_7 , E_8 , F_4 , G_2 , H_2 , H_3 , H_4 , or $I_2(m)$ for $m \geq 7$.

The following result shows that the coarse flag Hilbert–Poincaré series of (most) Coxeter arrangements may be viewed as “ Y -analogs” of the Hilbert series (1.15).

Theorem D. *Let \mathcal{A} be a Coxeter arrangement with no irreducible factor equivalent to E_8 and \mathbb{F} be a field. Then*

$$(1.16) \quad \frac{\text{cfHP}_{\mathcal{A}}(1, T)}{\pi_{\mathcal{A}}(1)} = \frac{E_{\text{rk}(\mathcal{A})}(T)}{(1-T)^{\text{rk}(\mathcal{A})}} = \text{Hilb}(\mathbb{F}[\text{sd}(\partial\Delta_{\text{rk}(\mathcal{A})-1})], T).$$

In other words,

$$\mathcal{N}_{\mathcal{A}}(1, T) = \pi_{\mathcal{A}}(1) E_{\text{rk}(\mathcal{A})}(T)$$

and, equivalently, for $1 \leq k \leq \text{rk}(\mathcal{A})$,

$$\sum_{\substack{F \in \Delta(\mathcal{L}(\mathcal{A})) \\ |F|=k-1}} \frac{\pi_F(1)}{\pi_{\mathcal{A}}(1)} = k! S(\text{rk}(\mathcal{A}), k).$$

The Stirling numbers of the second kind enter our proof of Theorem D via a simple formula, essentially due to Cayley, for the numbers of plane trees of a given length and number of leaves; see Lemma 6.12.

The simple examples in Section 6.2.2 show that the conclusion of Theorem D does not hold for general, non-Coxeter hyperplane arrangements, even when they are central: the coefficients of $\mathcal{N}_{\mathcal{A}}(1, T)$ are typically not multiples of $\pi_{\mathcal{A}}(1)$. However, equation (1.16) of Theorem D holds for small-rank non-Coxeter restrictions of type-D arrangements; cf. Appendix A.2.

Question 1.8. Which further hyperplane arrangements satisfy (1.16)? Which property of hyperplane arrangements does this equation reflect?

To prove Theorem D, we first reduce to the irreducible case by showing that coarse flag Hilbert–Poincaré series are, essentially, Hadamard multiplicative; see Proposition 6.3. For $X \in \{A, B, D\}$, the result is proven in Section 6.3; for the types $I_2(m)$ it follows from Proposition 4.2. We computed the coarse flag Hilbert–Poincaré series of the other irreducible Coxeter arrangement with the help of `HypIgu` [17], a SageMath [30] package developed by the first author to compute (coarse) flag Hilbert–Poincaré series and other rational functions associated with hyperplane arrangements. The results of these computations, along with many other examples,

are recorded in Appendix A; in each case, the validity of Theorem D follows by inspection. The type E_8 is excluded from Theorem D only because we do not supply a proof nor an explicit computation.

All our computations support the following general nonnegativity conjecture.

Conjecture E. *For all hyperplane arrangements \mathcal{A} , the polynomial $\mathcal{N}_{\mathcal{A}}(Y, T) \in \mathbb{Z}[Y, T]$ has nonnegative coefficients.*

Indeed, the polynomial $\mathcal{N}_{\mathcal{A}}(Y, T)$ has nonnegative coefficients for all of the arrangements in our appendix. We furthermore view Conjecture E as an extension of the following observation, which uses deep results from algebraic combinatorics. We note that $\pi_{\mathcal{A}}(Y)$ is the Poincaré polynomial of a quotient of an exterior algebra, known as the Orlik–Solomon algebra [19, Thm. 3.68].

Proposition 1.9. *For all hyperplane arrangements \mathcal{A} we have*

$$\mathcal{N}_{\mathcal{A}}(Y, 0) = \pi_{\mathcal{A}}(Y)$$

and, for all fields \mathbb{F} ,

$$\frac{\text{cfHP}_{\mathcal{A}}(0, T)}{1 - T} = \text{Hilb}(\mathbb{F}[\Delta(\mathcal{L}(\mathcal{A}))], T).$$

In particular, the coefficients of both $\mathcal{N}_{\mathcal{A}}(Y, 0)$ and $\mathcal{N}_{\mathcal{A}}(0, T)$ are nonnegative.

1.5. Notation. We let \mathbb{N} be the set of positive integers. For $I \subseteq \mathbb{N}$, we denote $I \cup \{0\}$ by I_0 . For $n \in \mathbb{N}$, set $[n] := \{1, \dots, n\}$. Let δ_P be 1 when property P is true and 0 otherwise. For a set I , denote by $\mathcal{P}(I)$ the power set of I and by $\mathcal{P}(I; k)$ the set of subsets of I of cardinality k . We set $\mathcal{P}(n) := \mathcal{P}([n])$ and $\mathcal{P}(n; k) := \mathcal{P}([n]; k)$. We set $\text{gp}(X) = X/(1 - X)$ and $\text{gp}_0(X) = 1/(1 - X)$.

Throughout, \mathbb{K} denotes a field of characteristic zero and K a number field with ring of integers \mathcal{O}_K . We write \mathfrak{o} for a compact discrete valuation ring (cdVR) of arbitrary characteristic, often assumed to be an \mathcal{O}_K -module. Its residue field has cardinality q and characteristic p . The following table records some further frequently used notation.

Symbol	Description	Reference
$\mathcal{A}, \mathcal{A}(\mathfrak{o})$	hyperplane arrangement in \mathbb{K}^d resp. in \mathfrak{o}^d	§ 1, 1.1
$\mathcal{L}(\mathcal{A})$	intersection poset of \mathcal{A}	§ 1
$\Delta(P)$	order complex of a poset P	§ 1
$\hat{0}, \hat{1}$	bottom resp. top elements of a poset	§ 1
\tilde{P}, \bar{P}	posets $P \setminus \{\hat{0}\}$ resp. $P \setminus \{\hat{0}, \hat{1}\}$	§ 1
X_n	Coxeter arrangement of type X and rank n	Eq. (5.1)
$\pi_{\mathcal{A}}(Y)$	Poincaré polynomial of \mathcal{A}	Eq. (1.1)
$\pi_F(Y)$	flag Poincaré polynomial of F	Eq. (1.3)
$\pi_F^{\circ}(Y), \bar{\pi}_F^{\circ}(Y)$	normalized Poincaré polynomials of F	Eq. (1.10)
$\text{fHP}_{\mathcal{A}}(Y, T)$	flag Hilbert–Poincaré series of \mathcal{A}	Def. 1.1
$\text{cfHP}_{\mathcal{A}}(Y, T)$	coarse flag Hilbert–Poincaré series of \mathcal{A}	Def. 1.7
$\mathcal{N}_{\mathcal{A}}(Y, T)$	numerator of $\text{cfHP}_{\mathcal{A}}(Y, T)$	Eq. (1.13)
$\zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s})$	analytic zeta function of $\mathcal{A}(\mathfrak{o})$	Def. 1.4
$\zeta_{\mathcal{A}}^{\text{top}}(\mathbf{s})$	multivariate topological zeta function of \mathcal{A}	Eq. (2.5)
$\zeta_{\mathcal{A}(\mathfrak{o})}^{\text{at}}(\mathbf{s})$	atom zeta function of $\mathcal{A}(\mathfrak{o})$	Def. 1.6
$Z_f(\mathbf{s})$	Igusa zeta function of $f \in \mathcal{O}_K[\mathbf{X}]$	Eq. (1.9)
$Z_f^{\text{top}}(\mathbf{s})$	topological zeta function of $f \in \mathcal{O}_K[\mathbf{X}]$	§ 1.2

$\mathcal{P}_X(I), \mathcal{P}_X(I; 2)$	Coxeter versions of $\mathcal{P}(I)$ resp. $\mathcal{P}(I; 2)$	§ 5.1
$\Pi_{X,n}$	set partitions of type X_n	§ 5.1
$\text{TP}_{X,n}, \text{TP}_{n+1}$	total partitions of type X_n resp. A_n	§ 5.2
UTP_{n+1}	unlabeled analog of TP_{n+1}	§ 5.3.1
$\pi_{X,\tau}(Y)$	Poincaré polynomial of X -labeled tree τ	Eq. (5.7)
$\text{Cgp}_{X,\tau}(Z, \mathbf{T})$	rational function of X -labeled tree τ	Eq. (5.8)
$\text{PT}_{n,k}$	plane trees with n leaves and k generations	Def. 6.11
$\text{LPT}_{X,n,k}, \text{LRT}_{X,n,k}$	labeled plane resp. rooted trees of type X_n	Def. 6.11
$\text{LPT}_{X,n,k}^0$	“0-labeled” plane trees of type X_n	Def. 6.16

2. ANALYTIC ZETA FUNCTIONS OF HYPERPLANE ARRANGEMENTS

Recall the assumptions made on the hyperplane arrangement \mathcal{A} at the beginning of Section 1.1. In particular, we assume that \mathcal{A} is defined over a number field K and all cDVRs \mathfrak{o} considered are \mathcal{O}_K -modules. Recall further Definition 1.4 of the analytic zeta function $\zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s})$ of \mathcal{A} over \mathfrak{o} , and that we write \mathbb{F}_q for the residue field of \mathfrak{o} . Theorem B will follow from the next theorem.

Theorem 2.1. *If \mathcal{A} has good reduction over \mathbb{F}_q , then*

$$(2.1) \quad \zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s}) = \sum_{F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))} \pi_F(-q^{-1}) \prod_{x \in F} \frac{q^{-\text{rk}(x) - \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y}}{1 - q^{-\text{rk}(x) - \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y}}.$$

If \mathcal{A} is also central, then

$$(2.2) \quad \zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s}) = \frac{1}{1 - q^{-\text{rk}(\mathcal{A}) - \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A})} s_y}} \sum_{F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))} \pi_F(-q^{-1}) \prod_{x \in F} \frac{q^{-\text{rk}(x) - \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y}}{1 - q^{-\text{rk}(x) - \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y}}.$$

Theorem B is now an immediate consequence: its first equation (1.7) follows from Theorem 2.1 and Definition 1.1 of $\text{fHP}_{\mathcal{A}}(Y, \mathbf{T})$. The second equation (1.8) is deduced from equation (1.7) by an application of Möbius inversion: setting $s_x = \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} (r_y - \text{rk}(y)) \mu(y, x)$, we have $r_x = \text{rk}(x) + \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y$.

In the remainder of this section, we prove Theorem 2.1. For the proof we will need the following lemma.

Lemma 2.2. *If \mathcal{A} has good reduction over \mathbb{F}_q , then*

$$(2.3) \quad \zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s}) = \sum_{x \in \mathcal{L}(\mathcal{A})} q^{-\text{rk}(x) - \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y} \pi_{\mathcal{A}^x}(-q^{-1}) \zeta_{\mathcal{A}_x(\mathfrak{o})} \left((s_y)_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} \right).$$

If \mathcal{A} is also central, then

$$(2.4) \quad \zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s}) = \frac{1}{1 - q^{-\text{rk}(\mathcal{A}) - \sum_{x \in \tilde{\mathcal{L}}(\mathcal{A})} s_x}} \sum_{x \in \mathcal{L}(\mathcal{A}) \setminus \{\hat{1}\}} q^{-\text{rk}(x) - \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y} \cdot \pi_{\mathcal{A}^x}(-q^{-1}) \zeta_{\mathcal{A}_x(\mathfrak{o})} \left((s_y)_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} \right).$$

Proof. Suppose \mathfrak{o} has maximal ideal \mathfrak{p} , and let $d = \dim(\mathcal{A})$. For $x \in \tilde{\mathcal{L}}(\mathcal{A})$, set

$$U_x = \{z \in \mathfrak{o}^d \mid L(z) \in \mathfrak{p} \iff L \in \mathcal{A}_x\}$$

and $U_{\hat{0}} = \mathfrak{o}^d \setminus \bigcup_{y \in \tilde{\mathcal{L}}(\mathcal{A})} U_y$. Fix $x \in \mathcal{L}(\mathcal{A})$. The number of vectors $v \in \mathbb{F}_q^d$ such that $v + \mathfrak{p}^d \subseteq U_x$ is $\chi_{\mathcal{A}^x}(q)$, where $\chi_{\mathcal{A}^x}(Y)$ is the characteristic polynomial of \mathcal{A}^x ; see

[28, Thm. 5.15]. Recall from (1.2) that $\chi_{\mathcal{A}^x}(Y) = Y^{d-\text{rk}(x)}\pi_{\mathcal{A}^x}(-Y^{-1})$. Since \mathcal{A} has good reduction over \mathbb{F}_q ,

$$\begin{aligned}\zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s}) &= \sum_{x \in \mathcal{L}(\mathcal{A})} \int_{U_x} \prod_{y \in \tilde{\mathcal{L}}(\mathcal{A})} \|\mathcal{A}_y\|^{s_y} |\mathrm{d}\mathbf{X}| \\ &= \sum_{x \in \mathcal{L}(\mathcal{A})} q^{-d-\sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y} \chi_{\mathcal{A}^x}(q) \zeta_{\mathcal{A}_x(\mathfrak{o})} \left((s_y)_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} \right) \\ &= \sum_{x \in \mathcal{L}(\mathcal{A})} q^{-\text{rk}(x)-\sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y} \pi_{\mathcal{A}^x}(-q^{-1}) \zeta_{\mathcal{A}_x(\mathfrak{o})} \left((s_y)_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} \right).\end{aligned}$$

We give some justification for the second equality: for each $x \in \mathcal{L}(\mathcal{A})$, we choose $v_x \in \mathbb{F}_q^d$ such that $v_x + \mathfrak{p}^d \subseteq U_x$ and apply a change of variables $X_i \mapsto v_{x,i} + \pi \tilde{X}_i$, where π is a uniformizer of \mathfrak{o} .

If \mathcal{A} is also central, then $\hat{1} \in \mathcal{L}(\mathcal{A})$, so

$$\int_{U_{\hat{1}}} \prod_{y \in \tilde{\mathcal{L}}(\mathcal{A})} \|\mathcal{A}_y\|^{s_y} |\mathrm{d}\mathbf{X}| = q^{-\text{rk}(\mathcal{A})-\sum_{y \in \tilde{\mathcal{L}}(\mathcal{A})} s_y} \zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s}).$$

Hence

$$\begin{aligned}\zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s}) &= q^{-\text{rk}(\mathcal{A})-\sum_{y \in \tilde{\mathcal{L}}(\mathcal{A})} s_y} \zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s}) + \\ &\quad \sum_{x \in \mathcal{L}(\mathcal{A}) \setminus \{\hat{1}\}} q^{-\text{rk}(x)-\sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y} \pi_{\mathcal{A}^x}(-q^{-1}) \zeta_{\mathcal{A}_x(\mathfrak{o})} \left((s_y)_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} \right).\end{aligned}$$

Solving for $\zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s})$ yields (2.4). \square

In Section 5.4, we will specify Lemma 2.2 combinatorially in case of the classical Coxeter arrangements.

2.1. Proof of Theorem B. As explained above, it suffices to prove Theorem 2.1. We start by proving (2.2) and thus assume that \mathcal{A} is central.

Observe that if $x \in \mathcal{L}(\mathcal{A})$ and $y \in \mathcal{L}(\mathcal{A}_x)$, then $y \leq x$. Thus, applying Lemma 2.2 recursively yields a sum indexed by a subset of $\Delta(\mathcal{L}(\mathcal{A}) \setminus \{\hat{1}\})$. With the only exception of $x = \hat{0}$, every term in the sum in Lemma 2.2 contains a $\zeta_{\mathcal{A}_x(\mathfrak{o})}$ -factor. Hence, applying Lemma 2.2 recursively yields a sum indexed by the flags in $\{F \in \Delta(\mathcal{L}(\mathcal{A})) \mid \hat{0} \in F, \hat{1} \notin F\}$, which is in bijection with $\Delta(\tilde{\mathcal{L}}(\mathcal{A}))$.

Let $F = (x_1 < \cdots < x_\ell) \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))$ be a (possibly empty) flag, and set $G = (x_0 < x_1 < \cdots < x_\ell)$ where $x_0 = \hat{0}$. We prove that the F -term in (2.2) is the sum given by applying Lemma 2.2 $\ell + 1$ times to G , starting with x_ℓ and descending to x_0 . By Lemma 2.2, the term associated with $y := x_\ell$ in (2.4) is

$$\frac{\pi_{\mathcal{A}^y}(-q^{-1})}{1 - q^{-\text{rk}(\mathcal{A})-\sum_{z \in \tilde{\mathcal{L}}(\mathcal{A})} s_z}} q^{-\text{rk}(y)-\sum_{z \in \tilde{\mathcal{L}}(\mathcal{A}_y)} s_z} \zeta_{\mathcal{A}_y(\mathfrak{o})} \left((s_z)_{z \in \tilde{\mathcal{L}}(\mathcal{A}_y)} \right).$$

If $\ell = 0$, then this is indeed equal to the F -term. Thus, by induction on ℓ , the G -term is

$$\frac{\prod_{k=0}^{\ell} \pi_{\mathcal{A}^{x_k}}(-q^{-1})}{1 - q^{-\text{rk}(\mathcal{A})-\sum_{z \in \tilde{\mathcal{L}}(\mathcal{A})} s_z}} \prod_{y \in F} \frac{q^{-\text{rk}(y)-\sum_{z \in \tilde{\mathcal{L}}(\mathcal{A}_y)} s_z}}{1 - q^{-\text{rk}(y)-\sum_{z \in \tilde{\mathcal{L}}(\mathcal{A}_y)} s_z}},$$

where $x_{\ell+1} = \emptyset$. Since this is the F -term in (2.2), equation (2.2) follows.

We proceed to the proof of (2.1). For each $x \in \tilde{\mathcal{L}}(\mathcal{A})$, the subarrangement \mathcal{A}_x is central, so by (2.2),

$$\zeta_{\mathcal{A}_x(\mathfrak{o})}(\mathbf{s}) =$$

$$\frac{1}{1 - q^{-\text{rk}(x) - \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y}} \sum_{F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}_x))} \pi_F(-q^{-1}) \prod_{y \in F} \frac{q^{-\text{rk}(y) - \sum_{z \in \tilde{\mathcal{L}}(\mathcal{A}_y)} s_z}}{1 - q^{-\text{rk}(y) - \sum_{z \in \tilde{\mathcal{L}}(\mathcal{A}_y)} s_z}}.$$

Substituting this expression into (2.3) yields

$$\begin{aligned} \zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s}) &= \sum_{x \in \mathcal{L}(\mathcal{A})} \frac{q^{-\text{rk}(x) - \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y} \pi_{\mathcal{A}^x}(-q^{-1})}{1 - q^{-\text{rk}(x) - \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y}} \\ &\quad \cdot \sum_{F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}_x))} \pi_F(-q^{-1}) \prod_{y \in F} \frac{q^{-\text{rk}(y) - \sum_{z \in \tilde{\mathcal{L}}(\mathcal{A}_y)} s_z}}{1 - q^{-\text{rk}(y) - \sum_{z \in \tilde{\mathcal{L}}(\mathcal{A}_y)} s_z}} \\ &= \sum_{F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))} \pi_F(-q^{-1}) \prod_{x \in F} \frac{q^{-\text{rk}(x) - \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y}}{1 - q^{-\text{rk}(x) - \sum_{y \in \tilde{\mathcal{L}}(\mathcal{A}_x)} s_y}}. \end{aligned}$$

This concludes the proof of Theorem 2.1 and thus Theorem B. \square

2.2. Topological zeta functions. Let Q and $\mathbf{s} = (s_x)_{x \in \tilde{\mathcal{L}}(\mathcal{A})}$ be indeterminates and abbreviate $(Q^{-s_x})_{x \in \tilde{\mathcal{L}}(\mathcal{A})}$ to $Q^{-\mathbf{s}}$. By expressing a rational function $W(Y, \mathbf{T}) \in \mathbb{Q}(Y, \mathbf{T})$ as a power series in $\mathbb{Q}(\mathbf{s})[[Q - 1]]$, via $W(Y, \mathbf{T}) \mapsto W(-Q^{-1}, Q^{-\mathbf{s}})$, one obtains a rational function $W^{\text{top}}(\mathbf{s}) \in \mathbb{Q}(\mathbf{s})$ as the constant term of the power series. Informally speaking, this yields the limit of $\zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s})$ as “ $q \rightarrow 1$.” For further details, see Denef–Loeser [11, Sec. 2] and Rossmann [22, Sec. 5].

For $a \in \mathbb{Z}[\mathbf{s}]$, note that $Q^a = (1 + (Q - 1))^a = \sum_{k=0}^{\infty} \binom{a}{k} (Q - 1)^k$, where $\binom{a}{k} = a(a-1)\cdots(a-k+1)/k!$. Thus, the constant term of $Q^a \in \mathbb{Q}(\mathbf{s})[[Q - 1]]$ is 1, and the constant term of

$$\frac{Q - 1}{Q^a - 1} = \left(\sum_{k=0}^{\infty} \binom{a}{k+1} (Q - 1)^k \right)^{-1}$$

is $1/a \in \mathbb{Q}(\mathbf{s})$.

For a flag $F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))$, recall definitions (1.10) of the rational functions $\pi_F^\circ(Y)$ and $\bar{\pi}_F^\circ(Y)$. The following lemma records the observation that they are actually often polynomials.

Lemma 2.3. *For a hyperplane arrangement \mathcal{A} and $F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))$, we have $\pi_F^\circ(Y) \in \mathbb{Z}[Y]$. If \mathcal{A} is central and $F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))$, then $\bar{\pi}_F^\circ(Y) \in \mathbb{Z}[Y]$.*

Proof. If an arrangement \mathcal{A}' is nonempty and central, then $\pi_{\mathcal{A}'}(Y)$ is divisible by $1 + Y$; see [19, Prop. 2.51]. Suppose that $F = (x_1 < \cdots < x_\ell) \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))$, with $\ell \geq 0$. Recalling that $x_0 = \hat{0}$, we find that

$$\pi_F^\circ(Y) = \pi_{\mathcal{A}^{x_\ell}}(Y) \prod_{k=1}^{\ell} \frac{\pi_{\mathcal{A}^{x_k}}(Y)}{1 + Y}.$$

For each $k \in [\ell]$, the arrangement \mathcal{A}^{x_k} is central and nonempty; thus $\pi_{\mathcal{A}^{x_k}}(Y) \in \mathbb{Z}[Y]$. If \mathcal{A} is central, then so is \mathcal{A}^{x_ℓ} . Moreover \mathcal{A}^{x_ℓ} is nonempty if and only if $\hat{1} \notin F$. Hence, $\bar{\pi}_F^\circ(Y) \in \mathbb{Z}[Y]$ if $F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))$. \square

We use the following notation in the next proof. For $f \geq 1$, let $\mathfrak{o}^{(f)}$ be the (essentially unique) unramified degree- f extension of \mathfrak{o} . As in [22, Def. 5.13], we take the following limit—well-defined for all cDVRs \mathfrak{o} whose residue characteristics avoid a finite set of rational primes, depending only on \mathcal{A} —as the definition of the *multivariate topological zeta function* associated with \mathcal{A} :

$$(2.5) \quad \zeta_{\mathcal{A}}^{\text{top}}(\mathbf{s}) := \lim_{\substack{f \rightarrow 0 \\ f \in \mathbb{N}}} \zeta_{\mathcal{A}(\mathfrak{o}^{(f)})}(\mathbf{s}).$$

Proof of Corollary 1.5. For $x \in \tilde{\mathcal{L}}(\mathcal{A})$, let $g_x(\mathbf{s})$ be as in (1.5). Then, by the above,

$$(2.6) \quad \begin{aligned} \text{fHP}_{\mathcal{A}}\left(-Q^{-1}, (Q^{-g_x(\mathbf{s})})_{x \in \tilde{\mathcal{L}}(\mathcal{A})}\right) &= \sum_{F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))} Q^{-|F|} \pi_F^{\circ}(-Q^{-1}) \prod_{x \in F} \frac{Q-1}{Q^{g_x(\mathbf{s})}-1} \\ &= \sum_{F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))} \pi_F^{\circ}(-1) \prod_{x \in F} \frac{1}{g_x(\mathbf{s})} + O((Q-1)). \end{aligned}$$

By Theorem B and [22, Thm. 5.12], $\zeta_{\mathcal{A}}^{\text{top}}(\mathbf{s})$ is the constant term in (2.6):

$$\zeta_{\mathcal{A}}^{\text{top}}(\mathbf{s}) = \lim_{\substack{f \rightarrow 0 \\ f \in \mathbb{N}}} \text{fHP}_{\mathcal{A}}\left(-q^{-f}, (q^{-f \cdot g_x(\mathbf{s})})_{x \in \tilde{\mathcal{L}}(\mathcal{A})}\right) = \sum_{F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))} \pi_F^{\circ}(-1) \prod_{x \in F} \frac{1}{g_x(\mathbf{s})}.$$

If \mathcal{A} is central, Lemma 2.3 implies that $\pi_F^{\circ}(-1) = 0$ for all $F \in \Delta(\bar{\mathcal{L}}(\mathcal{A}))$. Hence

$$\zeta_{\mathcal{A}}^{\text{top}}(\mathbf{s}) = \sum_{\substack{F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A})) \\ \hat{1} \in F}} \pi_F^{\circ}(-1) \prod_{x \in F} \frac{1}{g_x(\mathbf{s})} = \frac{1}{g_{\hat{1}}(\mathbf{s})} \sum_{F \in \Delta(\bar{\mathcal{L}}(\mathcal{A}))} \bar{\pi}_F^{\circ}(-1) \prod_{x \in F} \frac{1}{g_x(\mathbf{s})}. \quad \square$$

3. SELF-RECIPROCITY

In this section, we prove the self-reciprocity in Theorem A for $\text{fHP}_{\mathcal{A}}(Y, \mathbf{T})$ of a central hyperplane arrangement \mathcal{A} over a field of characteristic zero—without loss of generality, a number field K with ring of integers \mathcal{O}_K . For this we first prove the corresponding result (Corollary 3.2) for the analytic zeta function $\zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s})$ over generic cDVRs \mathfrak{o} , and then we apply Theorem B.

3.1. Functional equations for multivariate Igusa zeta functions. Let \mathcal{L} be a finite index set and $\mathcal{F} = (\mathbf{f}_x)_{x \in \mathcal{L}}$ with sets \mathbf{f}_x of polynomials over \mathcal{O}_K in d variables $\mathbf{X} = (X_1, \dots, X_d)$, each homogeneous of degree d_x . For a cDVR \mathfrak{o} with \mathcal{O}_K -module structure, set

$$(3.1) \quad Z_{\mathcal{F}(\mathfrak{o})}(\mathbf{s}) := \int_{\mathfrak{o}^d} \prod_{x \in \mathcal{L}} \|\mathbf{f}_x\|^{s_x} |d\mathbf{X}|.$$

As before, $|d\mathbf{X}|$ denotes the normalized additive Haar measure on \mathfrak{o}^d . The next theorem follows from well-known results.

Theorem 3.1 (Analytic self-reciprocity). *For such a family \mathcal{F} , there exists a finite set S of primes such that, if \mathfrak{o} is a cDVR and an \mathcal{O}_K -module with residue field of cardinality q and characteristic not in S , then the following holds:*

$$(3.2) \quad Z_{\mathcal{F}(\mathfrak{o})}(\mathbf{s}) \Big|_{q \rightarrow q^{-1}} = q^{-\sum_{x \in \mathcal{L}} d_x s_x} Z_{\mathcal{F}(\mathfrak{o})}(\mathbf{s}).$$

The proof of this local functional equation for a single homogeneous polynomial (i.e. $\mathcal{L} = \{x\}$ and $\mathbf{f}_x = \{f\}$) by Denef and Meuser [12] is based on an analysis of explicit formulae for the \mathfrak{p} -adic integral. In characteristic zero, the latter are obtained from a (Hironaka) resolution of singularities of the projective hypersurface defined by f . Veys and Zúñiga-Galindo noted that these formulae and arguments extend to the case of polynomial mappings (i.e. $|\mathcal{L}| = 1$); cf. [32]. The general case (i.e. finite \mathcal{L}) poses no additional conceptual difficulties; cf. [33]. At the cost of discarding finitely many further residue class characteristics, the transfer principle [8, Thm. 9.2.4] implies that the same formulae also hold in (sufficiently large) positive characteristic.

In general, the operation $q \rightarrow q^{-1}$ needs a precise algebro-geometric definition; see, e.g., [34, Rem. 1.7]. If, however, there exists $W \in \mathbb{Q}(Y, (T_x)_{x \in \mathcal{L}})$ such that, for all cDVRs \mathfrak{o} avoiding a finite set of “bad” residue characteristics,

$$Z_{\mathcal{F}(\mathfrak{o})}(\mathbf{s}) = W(-q^{-1}, (q^{-s_x})_{x \in \mathcal{L}}),$$

then the inversion of q amounts to the formal inversion of the variables Y and T_x , for each $x \in \mathcal{L}$; cf. [23, Sec. 4]. This is the case if $d_x = 1$ for all $x \in \mathcal{L}$, as in the current paper. The content of (3.2) is that W satisfies the palindromic symmetry

$$W(Y^{-1}, (T_x^{-1})_{x \in \mathcal{L}}) = \left(\prod_{x \in \mathcal{L}} T_x^{d_x} \right) W(Y, (T_x)_{x \in \mathcal{L}}).$$

Corollary 3.2. *Let \mathcal{A} be a central hyperplane arrangement defined over a number field K and \mathfrak{o} be a cDVR and an \mathcal{O}_K -module with residue field cardinality q such that \mathcal{A} has good reduction over \mathbb{F}_q . Then*

$$\zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s}) \Big|_{q \rightarrow q^{-1}} = q^{-\sum_{x \in \tilde{\mathcal{L}}(\mathcal{A})} s_x} \zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s}).$$

Proof. As \mathcal{A} is central we may, without loss of generality, assume that \mathcal{A} consists of homogeneous polynomials. Therefore $\zeta_{\mathcal{A}(\mathfrak{o})}(\mathbf{s})$ is of the form (3.1) with $d = \dim(\mathcal{A})$, $\mathcal{L} = \tilde{\mathcal{L}}(\mathcal{A})$, $\mathbf{f}_x = \mathcal{A}_x$, and $d_x = 1$ for all $x \in \tilde{\mathcal{L}}(\mathcal{A})$. Instead of choosing an unspecified (and uncontrollable) resolution of singularities of the hyperplane arrangement \mathcal{A} , we observe that $\tilde{\mathcal{L}}(\mathcal{A})$ is a prime example of what Hu [14] calls a simple arrangement of smooth subvarieties. For such arrangements a combinatorially defined chain of blow-ups along the respective worst-intersection locus yields a resolution of singularities [14, Thm. 1.1]. This resolution has good reduction over \mathbb{F}_q if and only if \mathcal{A} does. In this case, the resulting formulae apply to cDVRs of characteristic zero and positive characteristic alike. \square

3.2. Proof of Theorem A. Let \mathcal{A} , \mathfrak{o} , and q be as in Corollary 3.2. Recall the definition (1.6) of $h_x(\mathbf{r})$ for $x \in \tilde{\mathcal{L}}(\mathcal{A})$. By Möbius inversion we have $-\sum_{x \in \tilde{\mathcal{L}}(\mathcal{A})} h_x(\mathbf{r}) = \text{rk}(\hat{1}) - r_{\hat{1}} = \text{rk}(\mathcal{A}) - r_{\hat{1}}$. Hence, by Theorem B (1.8) and Corollary 3.2,

$$\begin{aligned} \text{fHP}_{\mathcal{A}}(-q, (q^{r_x})_{x \in \tilde{\mathcal{L}}(\mathcal{A})}) &= \zeta_{\mathcal{A}(\mathfrak{o})} \left((h_x(\mathbf{r}))_{x \in \tilde{\mathcal{L}}(\mathcal{A})} \right) \Big|_{q \rightarrow q^{-1}} \\ &= q^{-\sum_{x \in \tilde{\mathcal{L}}(\mathcal{A})} h_x(\mathbf{r})} \zeta_{\mathcal{A}(\mathfrak{o})} \left((h_x(\mathbf{r}))_{x \in \tilde{\mathcal{L}}(\mathcal{A})} \right) \\ &= q^{-\sum_{x \in \tilde{\mathcal{L}}(\mathcal{A})} h_x(\mathbf{r})} \text{fHP}_{\mathcal{A}}(-q^{-1}, (q^{-r_x})_{x \in \tilde{\mathcal{L}}(\mathcal{A})}) \\ &= q^{\text{rk}(\mathcal{A}) - r_{\hat{1}}} \text{fHP}_{\mathcal{A}}(-q^{-1}, (q^{-r_x})_{x \in \tilde{\mathcal{L}}(\mathcal{A})}). \end{aligned}$$

As this equality holds for infinitely many q , this proves Theorem A. \square

3.3. An alternative formulation. Writing $\text{fHP}_{\mathcal{A}}(Y, \mathbf{T})$ over the common denominator $\prod_{x \in \tilde{\mathcal{L}}(\mathcal{A})} (1 - T_x)$ and comparing coefficients, we see that the functional equation (1.4) is equivalent to the following consequence of Theorem A. For $\mathcal{S} \subseteq \overline{\mathcal{L}}(\mathcal{A})$, we set $\mathcal{S}^c = \overline{\mathcal{L}}(\mathcal{A}) \setminus \mathcal{S}$, both viewed as subposets of $\overline{\mathcal{L}}(\mathcal{A})$.

Corollary 3.3. *For all $\mathcal{S} \subseteq \overline{\mathcal{L}}(\mathcal{A})$,*

$$\sum_{F \in \Delta(\mathcal{S})} (-1)^{|F|} \pi_F(Y) = -(-Y)^{\text{rk}(\mathcal{A})} \sum_{F \in \Delta(\mathcal{S}^c)} (-1)^{|F|} \pi_F(Y^{-1}).$$

It is tempting to try and interpret this reciprocity result directly in terms of the topology and combinatorics of \mathcal{A} , bypassing the analytic reciprocity result Theorem 3.1. Such a proof might help to relax the assumptions of Theorem A to include, perhaps, central arrangements over fields of positive characteristic.

Remark 3.4. Corollary 3.3 for $\mathcal{S} = \overline{\mathcal{L}}(\mathcal{A})$ implies, in the case of a representable matroid, the identity in [16, Cor. 1.8].

4. EXAMPLES AND APPLICATIONS

We discuss various examples and applications of flag Hilbert–Poincaré series.

4.1. Arrangements of m parallel lines. Let $m \in \mathbb{N}$ and consider the arrangement $\mathcal{A} = \{H_1, \dots, H_m\} = \{X_2 - X_1 - k \mid k \in [m-1]_0\}$ of m parallel lines in \mathbb{K}^2 . Note that if $m = 1$, then $\mathcal{A} \cong \mathbf{A}_1$; if $m \geq 2$, then \mathcal{A} is not central. The flag Poincaré polynomials are

$$\pi_{\emptyset}(Y) = 1 + mY, \quad \pi_{(H_i)}(Y) = 1 + Y.$$

With these data, the following is evident.

Proposition 4.1. For $m \in \mathbb{N}$,

$$\begin{aligned} \text{fHP}_{\mathcal{A}}(Y, \mathbf{T}) &= 1 + mY + (1 + Y) \sum_{i=1}^m \frac{T_i}{1 - T_i}, \\ \zeta_{\mathcal{A}}^{\text{top}}(\mathbf{s}) &= 1 - m + \sum_{i=1}^m \frac{1}{1 + s_i}. \end{aligned}$$

From the proposition, it follows that

$$(4.1) \quad \text{fHP}_{\mathcal{A}}(Y^{-1}, (T_i^{-1})_{i=1}^m) = m(1 + Y^{-1}) - Y^{-1} \text{fHP}_{\mathcal{A}}(Y, \mathbf{T}).$$

If $m = 1$, then the right hand side of (4.1) is $-Y^{-1}T \text{fHP}_{\mathcal{A}}(Y, T)$. If $m \geq 2$, however, then $\text{fHP}_{\mathcal{A}}(Y, \mathbf{T})$ is not self-reciprocal.

4.2. Arrangements of m lines through the origin: $\mathbf{l}_2(m)$. Let $m \geq 2$ and ζ_m be a primitive m th root of unity. We consider the Coxeter arrangement $\mathcal{A} = \{H_1, \dots, H_m\}$ of type $\mathbf{l}_2(m)$, comprising m distinct lines H_i through the origin in K^2 , where $K = \mathbb{Q}(\zeta_m)$. Let \mathcal{A} be set of the m linear factors of the polynomial $X_1^m + X_2^m \in K[\mathbf{X}]$. The flag Poincaré polynomials are

$$\pi_{\emptyset}(Y) = (1 + Y)(1 + (m-1)Y), \quad \pi_{(H_i)}(Y) = (1 + Y)^2.$$

With these data, it is easy to deduce the following.

Proposition 4.2. For $m \geq 2$,

$$\begin{aligned} \text{fHP}_{\mathbf{l}_2(m)}(Y, \mathbf{T}) &= \frac{1 + Y}{1 - T_1} \left(1 + (m-1)Y + (1 + Y) \sum_{i=1}^m \frac{T_i}{1 - T_i} \right), \\ \zeta_{\mathbf{l}_2(m)}^{\text{top}}(\mathbf{s}) &= \frac{1}{2 + s_{\hat{1}} + s_1 + \dots + s_m} \left(2 - m + \sum_{i=1}^m \frac{1}{1 + s_i} \right). \end{aligned}$$

4.3. Shi arrangement \mathbf{SA}_2 . The Shi arrangement of type \mathbf{A}_2 is

$$\mathbf{SA}_2 = \{X_i - X_j - \varepsilon \mid 1 \leq i < j \leq 3, \varepsilon \in \{0, 1\}\}.$$

We write H_{ij} and H_{ij}^* for the affine subspaces determined by $X_i - X_j$ and $X_i - X_j - 1$ in \mathbb{K}^3 , respectively. These six (hyper-)planes intersect in six lines in \mathbb{K}^3 . Three of these lines are the intersection of three planes, three the intersections of two planes. Let $\mathbb{1}$ be the linear subspace of \mathbb{K}^3 spanned by $(1, 1, 1)$. The poset $\tilde{\mathcal{L}}(\mathbf{SA}_2)$ is given in Figure 4.1, and the flag Poincaré polynomials are

$$\begin{aligned} \pi_{\emptyset}(Y) &= (1 + 3Y)^2, \\ \pi_{(H_{12})}(Y) &= \pi_{(H_{23})}(Y) = \pi_{(H_{13}^*)}(Y) = (1 + Y)(1 + 2Y), \\ \pi_{(H_{13})}(Y) &= \pi_{(H_{12}^*)}(Y) = \pi_{(H_{23}^*)}(Y) = (1 + Y)(1 + 3Y), \end{aligned}$$

$$\begin{aligned}\pi_{(\mathbb{1})}(Y) &= \pi_{(\mathbb{1}+e_1)}(Y) = \pi_{(\mathbb{1}-e_3)}(Y) = (1+Y)(1+2Y), \\ \pi_{(\mathbb{1}+e_2)}(Y) &= \pi_{(\mathbb{1}-e_2)}(Y) = \pi_{(\mathbb{1}-e_2-2e_3)}(Y) = (1+Y)^2, \\ \pi_F(Y) &= (1+Y)^2,\end{aligned}$$

where $F \in \Delta(\tilde{\mathcal{L}}(\mathcal{SA}_2))$ is any maximal chain. Referring to Figure 4.1, we label the planes (resp. lines) from left to right by the integers $\{1, \dots, 6\}$ (resp. $\{7, \dots, 12\}$). Recall $\text{gp}(X)$ and $\text{gp}_0(X)$ are the respective geometric progressions $\frac{X}{1-X}$ and $\frac{1}{1-X}$.

Proposition 4.3. *We have*

$$\begin{aligned}\text{fHP}_{\mathcal{SA}_2}(Y, \mathbf{T}) &= (1+3Y)^2 + (1+Y)(1+2Y) \left(\text{gp}(T_1) + \text{gp}(T_3) + \text{gp}(T_5) + \text{gp}(T_7) \right. \\ &\quad \left. + \text{gp}(T_9) + \text{gp}(T_{11}) \right) + (1+Y)(1+3Y) \left(\text{gp}(T_2) + \text{gp}(T_4) \right. \\ &\quad \left. + \text{gp}(T_6) \right) + (1+Y)^2 \left(\text{gp}(T_8)(1 + \text{gp}(T_2) + \text{gp}(T_4)) \right. \\ &\quad \left. + \text{gp}(T_{10})(1 + \text{gp}(T_2) + \text{gp}(T_6)) + \text{gp}(T_{12})(1 + \text{gp}(T_4) + \text{gp}(T_6)) \right. \\ &\quad \left. + \text{gp}(T_7)(\text{gp}(T_1) + \text{gp}(T_2) + \text{gp}(T_3)) + \text{gp}(T_9)(\text{gp}(T_1) + \text{gp}(T_4)) \right. \\ &\quad \left. + \text{gp}(T_5) + \text{gp}(T_{11})(\text{gp}(T_3) + \text{gp}(T_5) + \text{gp}(T_6)) \right),\end{aligned}$$

$$\begin{aligned}\zeta_{\mathcal{SA}_2}^{\text{top}}(\mathbf{s}) &= 4 - \sum_{i=1}^6 \text{gp}_0(-s_i) - \sum_{i=1}^3 \text{gp}_0(-s_{2i}) \\ &\quad + \text{gp}_0(-s_1 - s_2 - s_3 - s_7 - 1) (\text{gp}_0(-s_1) + \text{gp}_0(-s_2) + \text{gp}_0(-s_3) - 1) \\ &\quad + \text{gp}_0(-s_1 - s_4 - s_5 - s_9 - 1) (\text{gp}_0(-s_1) + \text{gp}_0(-s_4) + \text{gp}_0(-s_5) - 1) \\ &\quad + \text{gp}_0(-s_3 - s_5 - s_6 - s_{11} - 1) (\text{gp}_0(-s_3) + \text{gp}_0(-s_5) + \text{gp}_0(-s_6) - 1) \\ &\quad + \text{gp}_0(-s_2 - s_4 - s_8 - 1) (\text{gp}_0(-s_2) + \text{gp}_0(-s_4)) \\ &\quad + \text{gp}_0(-s_2 - s_6 - s_{10} - 1) (\text{gp}_0(-s_2) + \text{gp}_0(-s_6)) \\ &\quad + \text{gp}_0(-s_4 - s_6 - s_{12} - 1) (\text{gp}_0(-s_4) + \text{gp}_0(-s_6)).\end{aligned}$$

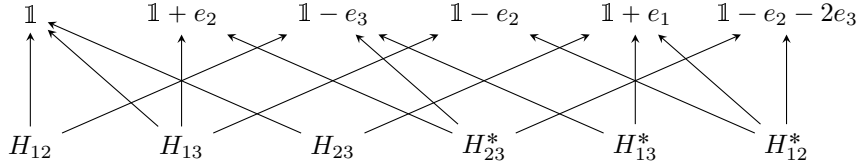


FIGURE 4.1. The poset $\tilde{\mathcal{L}}(\mathcal{SA}_2)$ for the Shi arrangement \mathcal{SA}_2 .

Formulae for the coarse flag Hilbert–Poincaré series associated with additional Shi arrangements are given in Section A.3.

4.4. Braid arrangement A_3 . The Coxeter (or braid) arrangement of type A_3 is

$$\{X_i - X_j \mid 1 \leq i < j \leq 4\}.$$

We write $H_{ij} = V(X_i - X_j)$. For distinct $i, j, k, \ell \in [4]$ and maximal flags F , the flag Poincaré polynomials are

$$\begin{aligned}\pi_{\emptyset}(Y) &= (1+Y)(1+2Y)(1+3Y), \\ \pi_{(H_{ij})}(Y) &= \pi_{(H_{ij} \cap H_{ik})}(Y) = (1+Y)^2(1+2Y), \\ \pi_{(H_{ij} \cap H_{k\ell})}(Y) &= \pi_F(Y) = (1+Y)^3.\end{aligned}$$

We associate to H_{ij} , $H_{ij} \cap H_{ik}$, and $H_{ij} \cap H_{k\ell}$ the indeterminates T_I , T_J , and $T_{I|K}$ and s_I , s_J , and $s_{I|K}$ respectively, provided $I = \{i, j\}$, $J = \{i, j, k\}$, and $K = \{k, \ell\}$.

Proposition 4.4.

$$\begin{aligned}
(1 - T_{\mathfrak{I}})\text{fHP}_{\mathbf{A}_3}(Y, \mathbf{T}) &= (1 + Y)(1 + 2Y)(1 + 3Y) \\
&+ (1 + Y)^2(1 + 2Y) \left(\sum_{I \in \mathcal{P}(4;2)} \text{gp}(T_I) + \sum_{J \in \mathcal{P}(4;3)} \text{gp}(T_J) \right) \\
&+ (1 + Y)^3 \left(\sum_{\substack{I, J \in \mathcal{P}(4;2) \\ I \cup J = [4]}} \text{gp}(T_{I|J}) (1 + \text{gp}(T_I) + \text{gp}(T_J)) \right) \\
&+ (1 + Y)^3 \left(\sum_{I \in \mathcal{P}(4;3)} \sum_{J \in \mathcal{P}(I;2)} \text{gp}(T_I) \text{gp}(T_J) \right).
\end{aligned}$$

With

$$Q(\mathbf{s}) = 3 + s_{\mathfrak{I}} + \sum_{I \in \mathcal{P}(4;2) \cup \mathcal{P}(4;3)} s_I + \sum_{\substack{I, J \in \mathcal{P}(4;2) \\ I \cup J = [4]}} s_{I|J},$$

$$\begin{aligned}
Q(\mathbf{s}) \cdot \zeta_{\mathbf{A}_3}^{\text{top}}(\mathbf{s}) &= 2 - \left(\sum_{I \in \mathcal{P}(4;2)} \text{gp}_0(-s_I) + \sum_{J \in \mathcal{P}(4;3)} \text{gp}_0 \left(-1 - s_J - \sum_{K \in \mathcal{P}(J;2)} s_K \right) \right) \\
&+ \left(\sum_{\substack{I, J \in \mathcal{P}(4;2) \\ I \cup J = [4]}} \text{gp}_0(-1 - s_{I|J} - s_I - s_J) (\text{gp}_0(-s_I) + \text{gp}_0(-s_J)) \right) \\
&+ \left(\sum_{I \in \mathcal{P}(4;3)} \sum_{J \in \mathcal{P}(I;2)} \text{gp}_0 \left(-1 - s_I - \sum_{K \in \mathcal{P}(I;2)} s_K \right) \text{gp}_0(-s_J) \right).
\end{aligned}$$

In Example 5.8 we obtain formulae for the Igusa and univariate topological zeta functions associated with \mathbf{A}_3 in terms of certain labeled rooted trees.

4.5. Boolean arrangements. Consider the arrangement \mathcal{A} comprising all the coordinate hyperplanes in \mathbb{K}^n , also known as the Boolean arrangement and equivalent to \mathbf{A}_1^n . Since \mathcal{A}_x^y is Boolean for all $x, y \in \mathcal{L}(\mathcal{A})$ with $y < x$, it follows that $\pi_F(Y) = (1 + Y)^n$ for all $F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))$. Thus,

$$\text{fHP}_{\mathbf{A}_1^n}(Y, \mathbf{T}) = (1 + Y)^n \sum_{F \in \Delta(\tilde{\mathcal{L}}(\mathcal{A}))} \prod_{x \in F} \frac{T_x}{1 - T_x}.$$

We identify $\Delta(\tilde{\mathcal{L}}(\mathcal{A}))$ with $\widetilde{\text{WO}}([n])$, the poset of chains of nonempty subsets of $[n]$ and rewrite $\text{fHP}_{\mathcal{A}}(Y, (T_I)_{I \in \mathcal{P}(n) \setminus \{\emptyset\}})$ in terms of the *weak order zeta function* (cf. [25, Def. 2.9])

$$(4.2) \quad \mathcal{I}_n^{\text{WO}}((T_I)_{I \in \mathcal{P}(n) \setminus \{\emptyset\}}) := \sum_{F \in \widetilde{\text{WO}}([n])} \prod_{I \in F} \frac{T_I}{1 - T_I};$$

Proposition 4.5. For $n \geq 1$,

$$\begin{aligned}
\text{fHP}_{\mathbf{A}_1^n}(Y, \mathbf{T}) &= (1 + Y)^n \mathcal{I}_n^{\text{WO}}(\mathbf{T}), \\
\zeta_{\mathbf{A}_1^n}^{\text{top}}(\mathbf{s}) &= \sum_{\substack{F \in \widetilde{\text{WO}}([n]) \\ |F|=n}} \prod_{I \in F} \frac{1}{|I| + \sum_{J \in \mathcal{P}(I) \setminus \{\emptyset\}} s_J}.
\end{aligned}$$

Proof. By Lemma 2.3, $\bar{\pi}_F^\circ(-1) = 0$ if and only if $F \in \widetilde{\text{WO}}([n])$ is not maximal. \square

The neat factorization of $\text{fHP}_{\mathbf{A}_1^n}(Y, \mathbf{T})$ as a product of a polynomial in Y and a rational function in \mathbf{T} seems to be atypical for these series. We have not observed such a factorization anywhere outside the family of Boolean arrangements.

4.6. Generic central arrangements. Let $m, n \in \mathbb{N}$ with $n \leq m$. We consider the arrangement $\mathcal{U}_{n,m}$ of m hyperplanes through the origin in \mathbb{K}^n in general position. That is, for each $k \leq n$, every k -set of hyperplanes intersects in a codimension- k subspace. This is also known as the m -element uniform matroid of rank n . Observe that $\mathcal{U}_{2,m} \cong \mathfrak{l}_2(m)$, covered in Section 4.2, and $\mathcal{U}_{n,n}$ is Boolean, seen in Section 4.5.

Lemma 4.6. *Let $m, n \in \mathbb{N}$ with $n \leq m$. The Poincaré polynomial of $\mathcal{U}_{m,n}(Y)$ is*

$$(4.3) \quad \pi_{\mathcal{U}_{n,m}}(Y) = (1+Y) \sum_{k=0}^{n-1} \binom{m-1}{k} Y^k.$$

Proof. If $n = m$, then $\pi_{\mathcal{U}_{n,n}}(Y) = (1+Y)^n$, and the lemma follows in this case, so we assume $m > n$. We proceed by induction on n , with base case $\mathcal{U}_{1,m} \cong \mathcal{U}_{1,1}$. Let $H \in \mathcal{U}_{n,m}$, so $\mathcal{U}_{n,m}^H \cong \mathcal{U}_{n-1,m-1}$ and $\mathcal{U}_{n,m} \setminus \{H\} \cong \mathcal{U}_{n,m-1}$. Hence by the Deletion-Restriction Theorem [19, Prop. 2.56],

$$(4.4) \quad \pi_{\mathcal{U}_{n,m}}(Y) = \pi_{\mathcal{U}_{n,m-1}}(Y) + Y \pi_{\mathcal{U}_{n-1,m-1}}(Y).$$

By induction, the lemma follows. \square

We augment our power-set notation to accommodate multiple, prescribed subset sizes. For $I, J \subset \mathbb{N}_0$, we define $\mathcal{P}(I; J) = \bigcup_{j \in J} \mathcal{P}(I; j) \subseteq \mathcal{P}(I)$. Observe that for $n \leq m$ we have $\bar{\mathcal{L}}(\mathcal{U}_{n,m}) \cong \mathcal{P}([m]; [n-1])$ and $\tilde{\mathcal{L}}(\mathcal{U}_{n,n}) \cong \mathcal{P}([n]; [n])$.

The next proposition generalizes Proposition 4.5, as $\mathbf{A}_1^n \cong \mathcal{U}_{n,n}$, and Proposition 4.2, since $\mathfrak{l}_2(m) \cong \mathcal{U}_{2,m}$.

Proposition 4.7. *Let $m, n \in \mathbb{N}$ with $n \leq m$. For $\mathbf{T} = (T_{\mathfrak{I}}, (T_I)_{I \in \mathcal{P}([m]; [n-1])})$ and $Q(\mathbf{s}) = n + s_{\mathfrak{I}} + \sum_{J \in \mathcal{P}([m]; [n-1])} s_J$,*

$$\begin{aligned} \text{fHP}_{\mathcal{U}_{n,m}}(Y, \mathbf{T}) &= \frac{1+Y}{1-T_{\mathfrak{I}}} \sum_{k=0}^{n-1} \binom{m-1}{k} Y^k + \sum_{\ell=1}^{n-1} \sum_{k=0}^{n-\ell-1} \frac{(1+Y)^{\ell+1}}{1-T_{\mathfrak{I}}} \binom{m-\ell-1}{k} Y^k \\ &\quad \cdot \sum_{I \in \mathcal{P}([m]; \ell)} T_I \mathcal{I}_\ell^{\text{WO}}((T_J)_{J \in \mathcal{P}(I; [\ell])}), \\ \zeta_{\mathcal{U}_{n,m}}^{\text{top}}(\mathbf{s}) \cdot Q(\mathbf{s}) &= \sum_{k=0}^{n-1} (-1)^k \binom{m-1}{k} \\ &\quad + \sum_{\ell=1}^{n-1} \sum_{I \in \mathcal{P}([m]; \ell)} \sum_{\substack{F \in \widetilde{\text{WO}}(I) \\ |F|=\ell}} \frac{\sum_{k=0}^{n-\ell-1} (-1)^k \binom{m-\ell-1}{k}}{\prod_{J \in F} (|J| + \sum_{K \in \mathcal{P}(J) \setminus \{\emptyset\}} s_K)}. \end{aligned}$$

Proof. For all $x \in \bar{\mathcal{L}}(\mathcal{U}_{n,m})$ with $\ell = \text{rk}(x)$, we find that $(\mathcal{U}_{n,m})_x \cong \mathbf{A}_1^\ell$ and $\mathcal{U}_{n,m}^x \cong \mathcal{U}_{n-\ell, m-\ell}$. Apply Theorem B, Lemmas 2.2 and 4.6, and Proposition 4.5, yielding the flag Hilbert–Poincaré series. For the topological zeta function, let $F = (x_1 < \cdots < x_\ell) \in \Delta(\bar{\mathcal{L}}(\mathcal{U}_{n,m}))$ with $\ell \geq 1$, and assume $\text{rk}(x_\ell) = r \in [n-1]$. If $r > \ell$, then $(x_1 < \cdots < x_{\ell-1})$ is not a maximal flag in $\mathcal{U}_{n,m}^{x_\ell} \cong \mathbf{A}_1^r$. Hence, $\bar{\pi}_F^\circ(-1) = 0$. Thus, we only consider flags $(x_1 < \cdots < x_\ell)$ such that $\text{rk}(x_i) = i$ for all $i \in [\ell]$. Applying Corollary 1.5, Proposition 4.5, and Lemma 4.6, the proposition follows. \square

Remark 4.8. Motivic zeta functions associated with generic (central) hyperplane arrangements are a focus of [6, Sec. 5 and 6]. We consider that the combinatorial

considerations there may yield descriptions of the *coefficients* of (specific substitutions of) the flag Hilbert–Poincaré series $\text{fHP}_{\mathcal{U}_{n,m}}$ described in Proposition 4.7.

4.7. The Fano plane – an example in characteristic two. Let \mathcal{A} be the Fano arrangement, comprising the seven (hyper-)planes in \mathbb{F}_2^3 . We compute $\text{fHP}_{\mathcal{A}}(Y, \mathbf{T})$ and verify that it satisfies (1.4) of Theorem A, despite the facts that \mathcal{A} is *a priori* an arrangement in characteristic two and has *a posteriori* no equivalent arrangement in characteristic zero; see [20, Prop. 6.4.8].

For $x \in \overline{\mathcal{L}}(\mathcal{A})$ and maximal flags $F \in \Delta(\overline{\mathcal{L}}(\mathcal{A}))$, the flag Poincaré polynomials are

$$\begin{aligned}\pi_{\emptyset}(Y) &= (1+Y)(1+2Y)(1+4Y), \\ \pi_{(x)}(Y) &= (1+Y)^2(1+2Y), \\ \pi_F(Y) &= (1+Y)^3.\end{aligned}$$

Define sets of integers:

$$\begin{aligned}\mathcal{X}_8 &= \{1, 3, 5\}, & \mathcal{X}_9 &= \{1, 4, 6\}, & \mathcal{X}_{10} &= \{1, 2, 7\}, & \mathcal{X}_{11} &= \{3, 4, 7\}, \\ \mathcal{X}_{12} &= \{5, 6, 7\}, & \mathcal{X}_{13} &= \{2, 4, 5\}, & \mathcal{X}_{14} &= \{2, 3, 6\}.\end{aligned}$$

Thus, for an appropriate labeling of the seven planes by $\{1, \dots, 7\}$ and the seven lines by $\{8, \dots, 14\}$, we obtain the following.

Proposition 4.9. For $\mathbf{T} = (T_{\hat{1}}, T_1, \dots, T_{14})$ and $Q(\mathbf{s}) = 3 + s_{\hat{1}} + s_1 + \dots + s_{14}$,

$$\begin{aligned}(1 - T_{\hat{1}})\text{fHP}_{\mathcal{A}}(Y, \mathbf{T}) &= (1+Y)(1+2Y)(1+4Y) + (1+Y)^2(1+2Y) \sum_{i=1}^{14} \text{gp}(T_i) \\ &\quad + (1+Y)^3 \sum_{i=8}^{14} \sum_{j \in \mathcal{X}_i} \text{gp}(T_i) \text{gp}(T_j), \\ Q(\mathbf{s})\zeta_{\mathcal{A}}^{\text{top}}(\mathbf{s}) &= 3 - \sum_{i=1}^7 \text{gp}_0(-s_i) \\ &\quad + \sum_{i=8}^{14} \text{gp}_0\left(-1 - s_i - \sum_{j \in \mathcal{X}_i} s_j\right) \left(-1 + \sum_{j \in \mathcal{X}_i} \text{gp}_0(-s_j)\right)\end{aligned}$$

One may verify with a computer algebra system, e.g. Maple¹, that

$$\text{fHP}_{\mathcal{A}}(Y^{-1}, T_{\hat{1}}^{-1}, T_1^{-1}, \dots, T_{14}^{-1}) = (-Y)^{-3} T_{\hat{1}} \text{fHP}_{\mathcal{A}}(Y, \mathbf{T}).$$

4.8. An application to ask zeta functions. We briefly explain a connection between the zeta functions we associate with Boolean arrangements (see Section 4.5) and zeta functions associated with hypergraphs introduced and studied in [24].

Hypergraphs are used to parametrize modules of $n \times m$ -matrices satisfying combinatorially defined support constraints. More precisely, if \mathbf{H} is a hypergraph on $[n]$ with hyperedges e_1, \dots, e_m , then let $M_{\mathbf{H}} \subseteq M_{n \times m}(\mathbb{Z})$ be the module of integral matrices $[a_{ij}]$ with $a_{ij} = 0$ whenever vertex i and hyperedge e_j are not incident. A hypergraph is uniquely determined by its *hyperedge multiplicities* $\boldsymbol{\mu} = (\mu_I)_{I \subseteq [n]} \in \mathbb{N}_0^{\mathcal{P}(n)}$, in particular, $m = \sum_{I \subseteq [n]} \mu_I$. For an arbitrary cDVR \mathfrak{o} with maximal ideal \mathfrak{p} of index q and $k \in \mathbb{N}_0$, we write $M_{\mathbf{H}}(\mathfrak{o}/\mathfrak{p}^k) := M_{\mathbf{H}} \otimes \mathfrak{o}/\mathfrak{p}^k$ and $M_{\mathbf{H}}(\mathfrak{o}) := M_{\mathbf{H}} \otimes \mathfrak{o}$. Informally speaking, these are the modules of $n \times m$ -matrices over the respective rings satisfying the support constraints defining $M_{\mathbf{H}}$.

¹Maple is a trademark of Waterloo Maple Inc.

The average size of the kernels in $M_{\mathbf{H}}(\mathfrak{o}/\mathfrak{p}^k)$ is

$$\text{ask}(M_{\mathbf{H}}(\mathfrak{o}/\mathfrak{p}^k)) = \frac{1}{|M_{\mathbf{H}}(\mathfrak{o}/\mathfrak{p}^k)|} \sum_{a \in M_{\mathbf{H}}(\mathfrak{o}/\mathfrak{p}^k)} |\ker(a)|.$$

The (analytic) ask zeta function of \mathbf{H} over \mathfrak{o} is

$$\zeta_{M_{\mathbf{H}}(\mathfrak{o})}^{\text{ask}}(s) = \sum_{k=0}^{\infty} \text{ask}(M_{\mathbf{H}}(\mathfrak{o}/\mathfrak{p}^k)) q^{-ks}.$$

(In the notation of [24, Sec. 3.2], the function $\zeta_{M_{\mathbf{H}}(\mathfrak{o})}^{\text{ask}}(s)$ coincides with the ask zeta function $\zeta_{\eta^{\circ}}^{\text{ask}}(s)$ associated with the incidence representation η° of \mathbf{H} over \mathfrak{o} .)

In [24, Thm. A], Rossmann and the second author prove the existence of a bivariate rational function $W_{\mathbf{H}}(X, T) \in \mathbb{Q}(X, T)$ such that $\zeta_{M_{\mathbf{H}}(\mathfrak{o})}^{\text{ask}}(s) = W_{\mathbf{H}}(q, q^{-s})$ for all \mathfrak{o} with residue field cardinality q . This rational function is given in terms of weak orders on $[n]$; cf. [24, Thm. C]. We show that these rational functions may be expressed as substitutions of analytic zeta functions associated with Boolean arrangements, which are determined by Proposition 4.5.

Since $\mathcal{L}(\mathbf{A}_1^{n+1})$ is isomorphic to the subset lattice of $[n+1]$, we index the indeterminates of $\zeta_{\mathbf{A}_1^{n+1}}$ in the next proposition by the nonempty subsets of $[n+1]$. We further define polynomials for each $J \subseteq [n+1]$,

$$f_{n,J}(X, Y, (Z_I)_{I \subseteq [n]}) := \begin{cases} X + Y - Z_{\emptyset} - n - 1 & \text{if } J = \{n+1\}, \\ -Z_{J \setminus \{n+1\}} & \text{if } \{n+1\} \subsetneq J, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.10. *Let \mathbf{H} be a hypergraph on $[n]$ with hyperedge multiplicities $\boldsymbol{\mu} = (\mu_I)_{I \subseteq [n]} \in \mathbb{N}_0^{\mathcal{P}(n)}$, and set $m = \sum_{I \subseteq [n]} \mu_I$. If \mathfrak{o} is a cDVR as above, then*

$$\zeta_{M_{\mathbf{H}}(\mathfrak{o})}^{\text{ask}}(s) = (1 - q^{-1})^{-1} \zeta_{\mathbf{A}_1^{n+1}} \left((f_{n,J}(s, m, (\mu_I)_{I \subseteq [n]})_{\emptyset \neq J \subseteq [n+1]}) \right).$$

Proof. In [24, § 5.1], Rossmann and the second author define the \mathfrak{p} -adic integral

$$Z_{[n], \emptyset}(s_0, (s_I)_{I \subseteq [n]}) = \int_{\mathfrak{o}^{n+1}} |Y|^{s_0} \prod_{I \subseteq [n]} \|Z_I; Y\|^{s_I} |d\mathbf{Z}| |dY|,$$

where $Z_I := \{Z_i \mid i \in I\}$, and one integrates with regards to the normalized additive Haar measure on \mathfrak{o}^{n+1} . Applying [24, (5.2)], we obtain

$$\begin{aligned} & \zeta_{\mathbf{A}_1^{n+1}(\mathfrak{o})} \left((f_{n,J}(s, m, (\mu_I)_{I \subseteq [n]})_{\emptyset \neq J \subseteq [n+1]}) \right) \\ &= \int_{\mathfrak{o}^{n+1}} \prod_{J \subseteq [n+1]} \|X_J\|^{f_{n,J}(s, m, (\mu_I)_{I \subseteq [n]})} |d\mathbf{X}| \\ &= \int_{\mathfrak{o}^{n+1}} |Y|^{s-n+m-1} \prod_{I \subseteq [n]} \|Z_I; Y\|^{-\mu_I} |d\mathbf{Z}| |dY| \\ &= Z_{[n], \emptyset}(s - n + m - 1, (-\mu_I)_{I \subseteq [n]}) \\ &= (1 - q^{-1}) \zeta_{M_{\mathbf{H}}(\mathfrak{o})}^{\text{ask}}(s). \end{aligned}$$

(For us, the invariant called d in [24, (5.2)] is zero.) \square

It is of interest to explore whether there are comparable applications of zeta functions associated with hyperplane arrangements other than Boolean ones, say to ask zeta functions associated with more general module representations.

5. CLASSICAL COXETER ARRANGEMENTS, TOTAL PARTITIONS, AND ROOTED TREES

We focus now on the classical Coxeter arrangements. In Section 5.4 we prove Theorem 5.6, the precise version of Theorem C. It provides a fully explicit, finitary combinatorial description of the atom zeta functions associated with these Coxeter arrangements in terms of total partitions or, equivalently, certain labeled rooted trees. The groundwork for this result is laid in Sections 5.1 and 5.2. Throughout this section, let $X \in \{A, B, D\}$ and $n \in \mathbb{N}$, with $n \geq 2$ if $X = D$.

5.1. Classical Coxeter arrangements and set partitions. We recall some standard facts and develop notation concerning classical Coxeter arrangements and their intersection posets. The classical Coxeter arrangements of rank n are

$$(5.1) \quad \begin{aligned} A_n &:= \{X_i - X_j \mid 1 \leq i < j \leq n+1\}, \\ B_n &:= \{X_i - \varepsilon X_j \mid 1 \leq i < j \leq n, \varepsilon \in \{-1, 1\}\} \cup \{X_k \mid k \in [n]\}, \\ D_n &:= \{X_i - \varepsilon X_j \mid 1 \leq i < j \leq n, \varepsilon \in \{-1, 1\}\}. \end{aligned}$$

We set $D_1 := A_1$, and $A_0 = B_0 = D_0$ is the empty arrangement.

The intersection posets of these hyperplane arrangements are isomorphic to the posets of set partitions of the respective types. In the sequel, we recall Björner and Wachs's [3] description of these posets.

The set $\Pi_{A,n}$ is the poset of set partitions of $[n+1]$, ordered by refinement, with minimal element $\hat{0} = \{\{1\}, \{2\}, \dots, \{n+1\}\}$ and maximal element $\hat{1} = \{[n+1]\}$. For example,

$$\Pi_{A,2} = \{0|1|2, 01|2, 02|1, 0|12, 012\}.$$

Here and below, we use the abbreviated notation such as $0|12$ instead of $\{\{0\}, \{1, 2\}\}$. We will refer to elements of set partitions as *blocks*.

The elements of $\Pi_{B,n}$ are obtained from set partitions of $[n]_0$, where the non-minimal integers of blocks not containing 0 may each be independently *barred*, i.e. decorated with a special symbol called *bar*. For example,

$$\Pi_{B,2} = \{0|1|2, 01|2, 02|1, 0|12, 0|\bar{1}2, 012\}.$$

We say that two (possibly barred) integers $i, j \in [n]_0$ have the *same parity* in $P \in \Pi_{B,n}$ if they are both barred or both not barred in P ; otherwise they have *different parity*. The *zero block* of $P \in \Pi_{B,n}$ is the unique block containing 0, denoted by P_0 . Before defining the poset structure on $\Pi_{B,n}$, we define $\Pi_{D,n} \subset \Pi_{B,n}$ as the sub(po-)set of elements whose zero block is not of size two. For example,

$$\Pi_{D,2} = \{0|1|2, 0|12, 0|\bar{1}\bar{2}, 012\}.$$

To define the poset structure on $\Pi_{B,n}$ and $\Pi_{D,n}$, we describe two operations on the blocks of a set partition in $\Pi_{B,n}$, viz. *bar* and *unbar*. To bar a block is to put a bar over all numbers without a bar (even the minimal one) and to remove the bar over all numbers with a bar. To unbar a block is to remove all bars. For example, if $b = \bar{1}\bar{2}\bar{3}\bar{4}$ is a block, then the bar and unbar of b are $\bar{b} = \bar{1}2\bar{3}4$ and $\tilde{b} = 1234$ respectively. By definition, partitions $\rho, \sigma \in \Pi_{B,n}$ satisfy $\rho \leq \sigma$ if, for each block b of ρ , either

- (1) b is contained in a nonzero block of σ ,
- (2) \bar{b} is contained in a nonzero block of σ , or
- (3) \tilde{b} is contained in the zero block of σ .

We recall the respective explicit isomorphisms $\Pi_{X,n} \cong \mathcal{L}(X_n)$ given in [3, Sec. 6–8]. For $i, j, k \in [n]$ with $i \neq j$, define hyperplanes $H_k = V(X_k)$ and

$$H_J = \begin{cases} V(X_i - X_j) & \text{if } J = \{i, j\} \text{ or } J = \{\bar{i}, \bar{j}\}, \\ V(X_i + X_j) & \text{if } J = \{\bar{i}, j\} \text{ or } J = \{i, \bar{j}\}. \end{cases}$$

Let $\rho_{A,n} : \Pi_{A,n} \rightarrow \mathcal{L}(A_n)$ be the isomorphism

$$(5.2) \quad P \mapsto \bigcap_{\substack{I \in P \\ |I| \geq 2}} \bigcap_{\substack{J \subseteq I \\ |J|=2}} H_J,$$

and let $\rho_{B,n} : \Pi_{B,n} \rightarrow \mathcal{L}(B_n)$ be the isomorphism

$$(5.3) \quad P \mapsto \left(\bigcap_{k \in P_0 \setminus \{0\}} H_k \right) \cap \bigcap_{\substack{I \in P \setminus \{P_0\} \\ |I| \geq 2}} \bigcap_{\substack{J \subseteq I \\ |J|=2}} H_J.$$

The isomorphism $\rho_{D,n} : \Pi_{D,n} \rightarrow \mathcal{L}(D_n)$ is obtained by restricting $\rho_{B,n}$ to $\Pi_{D,n}$.

For a set I , let $\mathcal{P}(I)$ be the power set of I . If $I \subset \mathbb{N}_0$, set $\mathcal{P}_A(I) := \mathcal{P}(I)$. Let $\mathcal{P}_B(I)$ be the set of subsets obtained from elements of $\mathcal{P}_A(I)$, where nonminimal elements of sets not containing 0 may be independently barred. For example,

$$\mathcal{P}_B(\{0, 1, 2\}) = \mathcal{P}_A(\{0, 1, 2\}) \cup \{\{1, \bar{2}\}\}.$$

Set $\mathcal{P}_D(I) = \mathcal{P}_B(I) \setminus \{\{0, i\} \mid 0 \neq i \in I\}$. We write $\mathcal{P}_X(I; 2)$ for the set of 2-element subsets of $\mathcal{P}_X(I)$, which are in bijection with the atoms of $\Pi_{X,n}$.

5.2. Total partitions and labeled rooted trees. A *total partition* (of type X_n) is a flag of partitions in $\Pi_{X,n}$ of the form

$$(5.4) \quad \hat{0} = P_0 < P_1 < \cdots < P_h < P_{h+1} = \hat{1}$$

such that $P_i \cap P_{i+1}$ is either empty or contains only singletons for all $i \in [h-1]$. The set of total partitions of type X is denoted by $\text{TP}_{X,n}$. We also write TP_{n+1} for $\text{TP}_{A,n}$. The sequence $(\#\text{TP}_n)_{n \in \mathbb{N}}$ is also known as the solution of Schröder's fourth problem; cf. [18, A000311].

Below we give a description of total partitions in terms of labeled rooted trees.

5.2.1. X_n -labeled rooted trees. To describe the sets $\text{TP}_{X,n}$ in terms of labeled rooted trees, we first establish some terminology, borrowing from [29, Appendix]. Given a rooted tree τ , we denote by $V(\tau)$ the set of vertices and by $P(\tau)$ the set of *parents* of τ , viz. vertices which are not leaves. A vertex u is a *descendant* of a vertex v if either u is a child of v or u is a descendant of a child of v . The term *ancestor* is defined analogously. Two distinct vertices are *siblings* if they have the same parent. We call a vertex u *unbranched* if either u is a leaf or u has exactly one leaf descendant. Given a labeled rooted tree τ , we denote by $v_\alpha := v_\alpha(\tau) \in V(\tau)$ the leaf of τ labeled α . Write $V(\tau, \alpha)$ for the set of all ancestors of v_α . We denote by $v_\alpha^+ := v_\alpha^+(\tau) \in V(\tau, \alpha)$ the first ancestor of v_α with at least two children.

Definition 5.1. Suppose that τ is a rooted tree with $n+1$ leaves. We call τ

- (1) *A-labeled* if each integer in $[n+1]$ labels a unique leaf of τ ,
- (2) *B-labeled* if each integer in $[n]_0$ labels a unique leaf of τ , where each positive integer may be barred,
- (3) *D-labeled* if τ is B-labeled and the following holds: if all of the children of v_0^+ are unbranched, then v_0^+ has at least three children.

We also say that τ is *X_n -labeled* if τ is X-labeled with $n+1$ leaves.

We write $\text{LRT}_{X,n}$ for the respective sets of X_n -labeled rooted trees.

Definition 5.2. Suppose τ is an \mathbf{X} -labeled rooted tree and $u \in V(\tau)$. If u is a leaf, let $\text{DLL}(\tau, u)$ be the (singleton) set containing the label of u ; otherwise, let $\text{DLL}(\tau, u)$ be the set of labels of all descendants of u which are leaves.

Definition 5.3. We call $\tau \in \text{LRT}_{\mathbf{X},n}$ in *standard form* if, for all $u \in V(\tau, 0)$ and for all children v of u , the minimal label of $\text{DLL}(\tau, v)$ is not barred. Note that every tree $\tau \in \text{LRT}_{\mathbf{A},n}$ is in standard form.

We now identify $\text{TP}_{\mathbf{X},n}$ with the subset of $\text{LRT}_{\mathbf{X},n}$ comprising trees in standard form with the property that every parent has at least two children. See, for example, Figure 5.1 for the set $\text{TP}_{\mathbf{B},2}$; the first, fourth, and fifth trees in Figure 5.1 form the set $\text{TP}_{\mathbf{D},2}$. See Figure 5.3 for an example tree in $\text{TP}_{\mathbf{B},9}$.

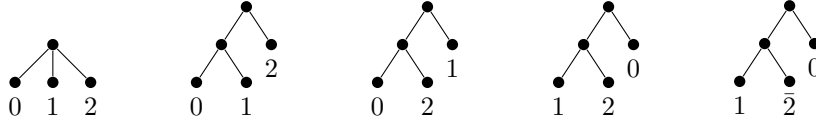


FIGURE 5.1. All of the trees in $\text{TP}_{\mathbf{B},2}$.

5.2.2. *From leaf labels to blocks.* Suppose that τ is an \mathbf{X}_n -labeled tree. The leaf labels of τ determine a unique block for each $u \in V(\tau)$, which we denote by $\lambda(\tau, u)$. These are the blocks that make up the set partitions in the flags. Below we describe how a labeled rooted tree $\tau \in \text{TP}_{\mathbf{X},n}$ uniquely determines a flag in $\Pi_{\mathbf{X},n}$ like in (5.4), but we emphasize that this works for all $\tau \in \text{LRT}_{\mathbf{X},n}$.

For a leaf v_α labeled α , we let $\lambda(\tau, v_\alpha)$ be the unbarred (singleton) block $\{\alpha\}$. If $u \in P(\tau)$ and $\mathbf{X} = \mathbf{A}$, then $\lambda(\tau, u)$ is the union of the $\lambda(\tau, u')$, where u' ranges over the children of u . This determines all blocks $\lambda(\tau, u)$ in type \mathbf{A} , so we assume that $\mathbf{X} \in \{\mathbf{B}, \mathbf{D}\}$. If $u \in V(\tau, 0)$, then $\lambda(\tau, u)$ is the union of the unbarred blocks $\lambda(\tau, u')$, where u' ranges over the children of u . If $u \in P(\tau) \setminus V(\tau, 0)$, then $\lambda(\tau, u)$ is the union of the $\lambda(\tau, u')$, where u' ranges over the children of u , each of which is barred, if necessary, so that the following hold.

- (1) For the unique child u' of u such that $\lambda(\tau, u')$ contains the minimal label in $\text{DLL}(\tau, u)$, the block $\lambda(\tau, u')$ is not barred, and
- (2) for all other children u'' of u , the block $\lambda(\tau, u'')$ is barred, if necessary, so the minimal labels in $\text{DLL}(\tau, u')$ and $\text{DLL}(\tau, u'')$ have the same parity (as labels) if and only if they have the same parity in $\lambda(\tau, u)$.

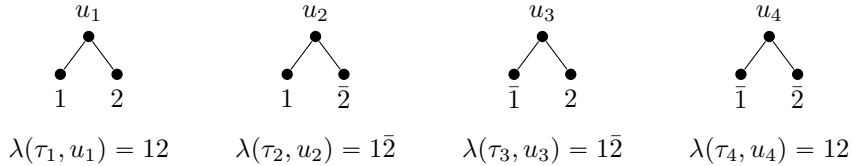


FIGURE 5.2. Four examples of blocks, contained in four different \mathbf{B} -labeled trees τ_i , uniquely determined by (1) and (2).

Figure 5.3 demonstrates the role of blocks in the translation between total partitions as flags and labeled rooted trees. With this identification, we will freely use tree-centric terminology in our further discussion of total partitions.

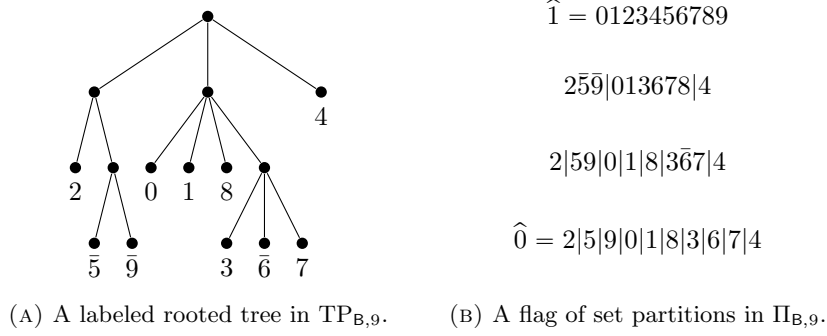


FIGURE 5.3. An illustration of the bijection between $\text{TP}_{\mathbf{B},9}$ and certain flags of set partitions in $\Pi_{\mathbf{B},9}$.

5.3. Atom zeta functions and rooted trees. Theorem C is a paraphrase of the more precise Theorem 5.6, which gives uniform formulae for the atom zeta function $\zeta_{\mathbf{X}_n(o)}^{\text{at}}(\mathbf{s})$, for $\mathbf{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{D}\}$, in terms of \mathbf{X}_n -labeled rooted trees. We prove Theorem 5.6 in Section 5.4. In Corollary 5.7 we record a variant of this formula for Igusa’s local zeta function associated with the braid arrangements in terms of unlabeled trees. In Corollary 5.9 we give formulae for types B and D in terms of total partitions of type A.

We begin by giving definitions of the numerical data in the statement of Theorem C. For a rooted tree τ and $u \in V(\tau)$, we define the counting functions

$$(5.5) \quad c(\tau, u) = \text{the number of children of } u,$$

$$(5.6) \quad u(\tau, u) = \text{the number of unbranched children of } u.$$

If $\tau \in \text{TP}_{\mathbf{X},n}$, then $c(\tau, u) \neq 1$ for all $u \in V(\tau)$.

For $n \geq 0$, define polynomials $((n))_Y = 1 + nY$ and

$$((n))_Y! = \prod_{i=1}^n ((i))_Y, \quad ((n))_Y!! = \prod_{i=1}^n ((2i-1))_Y.$$

Note that $((0))_Y! = ((0))_Y!! = 1$. For $\tau \in \text{LRT}_{\mathbf{X},n}$, we define three polynomials in $\mathbb{Z}[Y]$, depending on the type $\mathbf{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{D}\}$:

$$(5.7) \quad \begin{aligned} \pi_{\mathbf{A},\tau}(Y) &= \prod_{u \in P(\tau)} ((c(\tau, u) - 1))_Y!, \\ \pi_{\mathbf{B},\tau}(Y) &= \left(\prod_{u \in V(\tau,0)} ((c(\tau, u) - 1))_Y!! \right) \left(\prod_{u \in P(\tau) \setminus V(\tau,0)} ((c(\tau, u) - 1))_Y! \right), \\ \pi_{\mathbf{D},\tau}(Y) &= \pi_{\mathbf{B},\tau}(Y) \cdot \frac{((2c(\tau, v_0^+(\tau)) - u(\tau, v_0^+(\tau)) - 2))_Y ((c(\tau, v_0^+(\tau)) - 2))_Y!!}{((c(\tau, v_0^+(\tau)) - 1))_Y!!}; \end{aligned}$$

see, for instance, Examples 5.4 and 5.5. In fact, $\pi_{\mathbf{X},\tau}(Y)$ is the Poincaré polynomial $\pi_F(Y)$, where F is a flag determined by \mathbf{X} and τ ; cf. Lemma 6.14.

Example 5.4 (Poincaré polynomials of classical Coxeter arrangements). Let τ be the rooted tree whose root vertex has exactly $n+1$ children, all of which are leaves. For each $\mathbf{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{D}\}$, there is a unique \mathbf{X} -labeling for τ . The polynomials $\pi_{\mathbf{X},\tau}(Y)$ are just the Poincaré polynomials $\pi_{\mathbf{X}_n}(Y)$ of the respective arrangements:

$$\pi_{\mathbf{A},\tau}(Y) = \prod_{i=1}^n (1 + iY) = \pi_{\mathbf{A}_n}(Y),$$

$$\begin{aligned}\pi_{\mathbf{B},\tau}(Y) &= \prod_{i=1}^n (1 + (2i-1)Y) = \pi_{\mathbf{B}_n}(Y), \\ \pi_{\mathbf{D},\tau}(Y) &= (1 + (n-1)Y) \prod_{i=1}^{n-1} (1 + (2i-1)Y) = \pi_{\mathbf{D}_n}(Y);\end{aligned}$$

cf. [19, Thm. 4.137].

Suppose that τ is an \mathbf{X} -labeled rooted tree with root vertex v . Let $V_\circ(\tau) = P(\tau) \setminus \{v\}$ and $V_\circ(\tau, \alpha) = V(\tau, \alpha) \setminus \{v\}$. Recalling that $\text{gp}(X)$ denotes the geometric progression $\frac{X}{1-X}$, we define the geometric progression

$$(5.8) \quad \begin{aligned}\text{Cgp}_{\mathbf{X},\tau}(Z, \mathbf{T}) &= \left(\prod_{u \in V_\circ(\tau) \setminus V_\circ(\tau, 0)} \text{gp} \left(Z^{|\lambda(\tau, u)|-1} \prod_{J \in \mathcal{P}(\lambda(\tau, u); 2)} T_J \right) \right) \\ &\cdot \left(\prod_{u \in V_\circ(\tau, 0)} \text{gp} \left(Z^{|\lambda(\tau, u)|-1} \prod_{J \in \mathcal{P}_X(\lambda(\tau, u); 2)} T_J \right) \right).\end{aligned}$$

If $\mathbf{X} = \mathbf{A}$, then the second factor is 1 since no leaf is labeled with 0. Examples of $\text{Cgp}_{\mathbf{X},\tau}(Z, \mathbf{T})$ are seen in Example 5.5.

Example 5.5 (Numerical data). Figure 5.4 shows the numerical data associated with the tree $\tau \in \text{TP}_{\mathbf{B},9}$ from Figure 5.3, where

$$\begin{aligned}C_{A_1} &= \text{gp}(Z T_{59}), & C_{A_2} &= \text{gp}(Z^2 T_{25} T_{29} T_{59}), \\ C'_{A_2} &= \text{gp}(Z^2 T_{36} T_{37} T_{67}), & C_{B_5} &= \text{gp} \left(Z^5 \prod_{J \in \mathcal{P}_{\mathbf{B}}(\{0,1,3,6,7,8\}; 2)} T_J \right).\end{aligned}$$

Thus, $\text{Cgp}_{\mathbf{B},\tau}(Z, \mathbf{T}) = C_{A_1} C_{A_2} C'_{A_2} C_{B_5}$ and $\pi_{\mathbf{B},\tau}(Y) = ((2))_Y!! ((3))_Y!! ((1))_Y!^2 ((2))_Y!$.

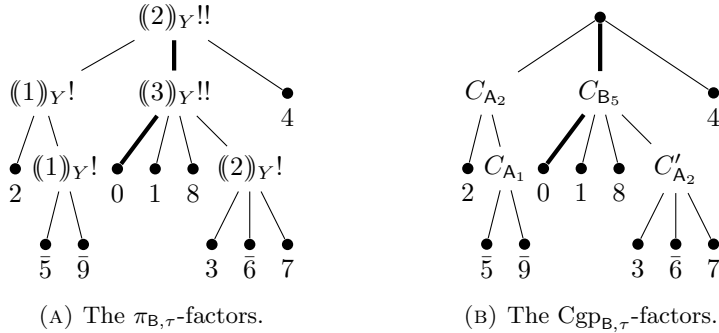


FIGURE 5.4. The tree $\tau \in \text{TP}_{\mathbf{B},9}$ from Figure 5.3 revisited.

Theorem 5.6 (Theorem C made precise). *Let $\mathbf{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{D}\}$ and $n \in \mathbb{N}$, with $n \geq 2$ if $\mathbf{X} = \mathbf{D}$. For all $c\text{DVR } \mathfrak{o}$ with residue field cardinality q , assumed to be odd unless $\mathbf{X} = \mathbf{A}$,*

$$\begin{aligned}\zeta_{\mathbf{X}_n(\mathfrak{o})}^{\text{at}} \left(((s_J)_{J \in \mathcal{P}_X(n;2)}) \right) &= \\ \text{gp}_0 \left(q^{-n - \sum_{J \in \mathcal{P}_X(n;2)} s_J} \right) &\sum_{\tau \in \text{TP}_{\mathbf{X},n}} \pi_{\mathbf{X},\tau}(-q^{-1}) \cdot \text{Cgp}_{\mathbf{X},\tau} \left(q^{-1}, (q^{-s_J})_{J \in \mathcal{P}_X(\lambda(\tau);2)} \right).\end{aligned}$$

5.3.1. *Corollaries of Theorem C.* Let UTP_{n+1} be the set of unlabeled rooted trees with $n + 1$ leaves where every parent has at least two children. (As the notation suggests, UTP_{n+1} is obtained from TP_{n+1} by removing the labels.) The number of distinct \mathbf{A} -labelings for $\tau \in \text{UTP}_{n+1}$ is $(n + 1)!/|\text{Aut}(\tau)|$, where $\text{Aut}(\tau)$ is the subgroup of the graph automorphism group of τ stabilizing the root. The sequence $(\#\text{UTP}_n)_{n \in \mathbb{N}}$ is [18, A000669]. For all $\tau \in \text{UTP}_{n+1}$, if τ' and τ'' are two \mathbf{A} -labelings of τ , then $\pi_{\mathbf{A},\tau'}(Y) = \pi_{\mathbf{A},\tau''}(Y)$, so we set $\pi_\tau(Y) := \pi_{\mathbf{A},\tau'}(Y)$. For $\tau \in \text{UTP}_{n+1}$ and $u \in V(\tau)$, let $\text{dl}(\tau, u)$ be the number of descendants of u , including u itself, that are leaves. Recall that $f_{\mathbf{A}_n}(\mathbf{X}) = \prod_{1 \leq i < j \leq n+1} (X_i - X_j)$.

Corollary 5.7. *For all cDVR \mathfrak{o} with residue field cardinality q , the Igusa local zeta function associated with $f_{\mathbf{A}_n}$ over \mathfrak{o} is*

$$Z_{f_{\mathbf{A}_n}, \mathfrak{o}}(s) = \text{gp}_0 \left(q^{-n - \binom{n+1}{2}s} \right) \sum_{\tau \in \text{UTP}_{n+1}} \frac{(n+1)!}{|\text{Aut}(\tau)|} \pi_\tau(-q^{-1}) \prod_{u \in V_\circ(\tau)} \text{gp} \left(q^{1 - \text{dl}(\tau, u) - \binom{\text{dl}(\tau, u)}{2}s} \right).$$

Example 5.8 (Braid arrangement \mathbf{A}_3). Consider the braid arrangement \mathbf{A}_3 , with $f_{\mathbf{A}_3}(\mathbf{X}) = \prod_{1 \leq i < j \leq 4} (X_i - X_j)$. All of the numerical data can be read off from the five trees comprising the set UTP_4 , listed in Figure 5.5. Specifically we find that

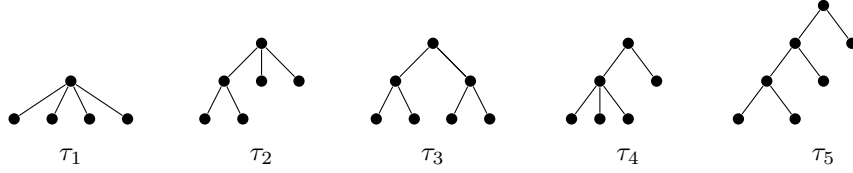


FIGURE 5.5. The five rooted trees in UTP_4 .

$$\begin{aligned} \text{Aut}(\tau_1) &= S_4, & \text{Aut}(\tau_2) &= S_2 \times S_2, & \text{Aut}(\tau_3) &= S_2 \wr S_2, \\ \text{Aut}(\tau_4) &= S_3, & \text{Aut}(\tau_5) &= S_2, \end{aligned}$$

$$\begin{aligned} \pi_{\tau_1}(Y) &= (1+Y)(1+2Y)(1+3Y), & \pi_{\tau_2}(Y) &= \pi_{\tau_4}(Y) = (1+Y)^2(1+2Y), \\ \pi_{\tau_3}(Y) &= \pi_{\tau_5}(Y) = (1+Y)^3. \end{aligned}$$

By Corollary 5.7, for all cDVR \mathfrak{o} with residue field cardinality q , Igusa's zeta function associated with $f := f_{\mathbf{A}_3}$ is

$$\begin{aligned} Z_{f, \mathfrak{o}}(s) &= \frac{1 - q^{-1}}{1 - q^{-3-6s}} \left((1 - 2q^{-1})(1 - 3q^{-1}) \right. \\ &\quad + \frac{6q^{-1-s}(1 - q^{-1})(1 - 2q^{-1})}{1 - q^{-1-s}} + \frac{3q^{-2-2s}(1 - q^{-1})^2}{(1 - q^{-1-s})^2} \\ &\quad \left. + \frac{4q^{-2-3s}(1 - q^{-1})(1 - 2q^{-1})}{1 - q^{-2-3s}} + \frac{12q^{-3-4s}(1 - q^{-1})^2}{(1 - q^{-1-s})(1 - q^{-2-3s})} \right). \end{aligned}$$

It might be instructive to compare this formula with the formula for $\text{fHP}_{\mathbf{A}_3}$ given in Proposition 4.4. Arguing as in Section 2.2, we obtain a formula for the topological zeta function of f in terms of the five trees in UTP_4 :

$$\begin{aligned} Z_f^{\text{top}}(s) &= \frac{1}{3+6s} \left(2 - \frac{6}{1+s} + \frac{3}{(1+s)^2} - \frac{4}{2+3s} + \frac{12}{(1+s)(2+3s)} \right) \\ &= \frac{2 - s - 2s^2 + 2s^3}{(1+s)^2(1+2s)(2+3s)}. \end{aligned}$$

Turning now to atom zeta functions associated with Coxeter arrangements of type B, we define an embedding $\varphi : \text{TP}_{\mathbf{A},n} \rightarrow \text{TP}_{\mathbf{B},n}$ by replacing the label $n+1$ with the label 0. Thus, the image of φ comprises all the trees whose labels have no bars, so each tree in the image is in standard form. For $\tau \in \text{LRT}_{\mathbf{B},n}$, set

$$\text{bars}(\tau) := 2^n \prod_{u \in V(\tau,0)} 2^{1-c(\tau,u)} \in \mathbb{N}.$$

Corollary 5.9. *For all $n \geq 1$ and $c\text{DVR } \mathfrak{o}$ with residue field of odd cardinality q ,*

$$\begin{aligned} \zeta_{\mathbf{B}_n(\mathfrak{o})}^{\text{at}}((s_J)_{J \in \mathcal{P}_{\mathbf{B}}(n;2)}) &= \mathbf{gp}_0 \left(q^{-n - \sum_{J \in \mathcal{P}_{\mathbf{B}}(n;2)} s_J} \right) \\ &\cdot \sum_{\tau \in \varphi(\text{TP}_{\mathbf{A},n})} \text{bars}(\tau) \pi_{\mathbf{B},\tau}(-q^{-1}) \mathbf{Cgp}_{\mathbf{B},\tau} \left(q^{-1}, (q^{-s_J})_{J \in \mathcal{P}_{\mathbf{B}}(\lambda(\tau);2)} \right), \end{aligned}$$

$$\begin{aligned} \zeta_{\mathbf{D}_n(\mathfrak{o})}^{\text{at}}((s_J)_{J \in \mathcal{P}_{\mathbf{D}}(n;2)}) &= \mathbf{gp}_0 \left(q^{-n - \sum_{J \in \mathcal{P}_{\mathbf{D}}(n;2)} s_J} \right) \\ &\cdot \sum_{\tau \in \varphi(\text{TP}_{\mathbf{A},n}) \cap \text{TP}_{\mathbf{D},n}} \text{bars}(\tau) \pi_{\mathbf{D},\tau}(-q^{-1}) \mathbf{Cgp}_{\mathbf{D},\tau} \left(q^{-1}, (q^{-s_J})_{J \in \mathcal{P}_{\mathbf{D}}(\lambda(\tau);2)} \right). \end{aligned}$$

Proof. Let $\beta : \text{TP}_{\mathbf{B},n} \rightarrow \text{TP}_{\mathbf{A},n}$ be the map given by removing bars from all leaves and replacing the label 0 with $n+1$. Thus $\beta \circ \varphi$ is the identity map on $\text{TP}_{\mathbf{A},n}$. From Definition 5.3, $|\beta^{-1}(\tau)| = \text{bars}(\tau)$. Since $\text{TP}_{\mathbf{D},n} \subset \text{TP}_{\mathbf{B},n}$, the corollary follows. \square

5.4. Proof of Theorem C. We specify Lemma 2.2 for the three specific families of Coxeter arrangements, also providing combinatorial reinterpretations. We use the three resulting Lemmas 5.14, 5.15, and 5.16 to recursively prove Theorem 5.6, and thus also Theorem C. We begin, however, with two key lemmas concerning the restrictions of classical Coxeter arrangements.

Lemma 5.10. *Let $X \in \{\mathbf{A}, \mathbf{B}\}$ and $n \in \mathbb{N}$. If $x \in \mathcal{L}(X_n)$, then $X_n^x \cong X_{n-\text{rk}(x)}$.*

Proof. The idea is the same for both types; we spell it out for type B. Set $P = \rho_{\mathbf{B},n}^{-1}(x) \in \Pi_{\mathbf{B},n}$, where $\rho_{\mathbf{B},n}$ is as in (5.3). It follows that $\text{rk}(x) = n - |P| + 1$. Fix $I, J, K \in P \setminus \{P_0\}$ with $I \neq J$ and $\varepsilon \in \{-1, 1\}$, and unbar I, J , and K . Then, for all $i \in I$ and $j \in J$,

$$x \cap V(X_i - \varepsilon X_j) = x \cap V(X_{\min(I)} - \varepsilon X_{\min(J)}).$$

Furthermore, for all $k \in K$, $\ell \in P_0 \setminus \{0\}$, and $\lambda \in \{-1, 0, 1\}$,

$$x \cap V(X_k - \lambda X_\ell) = x \cap V(X_{\min(K)}).$$

Let $\{X_I \mid I \in P \setminus \{P_0\}\}$ be independent variables. Then

$$\mathbf{B}_n^x \cong \{X_I \pm X_J \mid I, J \in P \setminus \{P_0\}, I \neq J\} \cup \{X_K \mid K \in P \setminus \{P_0\}\} \cong \mathbf{B}_{n-\text{rk}(x)}. \quad \square$$

The case for type D is not as simple as Lemma 5.10 is for types A and B, but follows similar reasoning. In fact, some restrictions in \mathbf{D}_n are not even Coxeter arrangements, and these non-Coxeter restrictions are the subject of Lemma 5.13.

Definition 5.11. For $n \in \mathbb{N}$ and $m \in [n]_0$, let

$$\mathcal{A}_{n,m} := \mathbf{D}_n \cup \{X_k \mid k \in [n-m]\}.$$

Lemma 5.12. *Let $n \geq 2$ and $x \in \mathcal{L}(\mathbf{D}_n)$. If there exists $k \in [n]$ such that $x \subseteq V(X_k)$, then $\mathbf{D}_n^x \cong \mathbf{B}_{n-\text{rk}(x)}$. Otherwise, $\mathbf{D}_n^x \cong \mathcal{A}_{n-\text{rk}(x),m}$, for some $m \in [n]_0$.*

Proof. Set $P = \rho_{\mathbf{D},n}^{-1}(x) \in \Pi_{\mathbf{D},n}$, where $\rho_{\mathbf{D},n}$ is the restriction of $\rho_{\mathbf{B},n}$ defined in (5.3). We distinguish two cases: either there exists $k \in [n]$ such that $x \subseteq V(X_k)$ or not.

In the first case, if such a k exists, then $|P_0| \geq 3$. Let $K \in P \setminus \{P_0\}$ and $\varepsilon \in \{-1, 1\}$, and unbar K . So, for all $k \in K$ and $\ell \in P_0 \setminus \{0\}$,

$$x \cap V(X_k - \varepsilon X_\ell) = x \cap V(X_{\min(K)} - \varepsilon X_{\max(P_0)}).$$

Thus, similar to Lemma 5.10, with independent variables $\{X_I \mid I \in P \setminus \{P_0\}\}$,

$$(5.9) \quad \mathcal{D}_n^x \cong \{X_I \pm X_J \mid I, J \in P \setminus \{P_0\}, I \neq J\} \cup \{X_K \mid K \in P \setminus \{P_0\}\} \cong \mathcal{B}_{n-\text{rk}(x)}.$$

In the second case, if no such k exists, then $P_0 = \{0\}$. Let $I, J, K \in P \setminus \{P_0\}$ with $I \neq J$ and $\varepsilon \in \{-1, 1\}$, and unbar I, J , and K . For all $i \in I$ and $j \in J$,

$$x \cap V(X_i - \varepsilon X_j) = x \cap V(X_{\min(I)} - \varepsilon X_{\min(J)}).$$

If $|K| \geq 2$, then, for any two distinct $k, k' \in K$, the element x is contained in either $V(X_k - X_{k'})$ or $V(X_k + X_{k'})$ but not both since $K \neq P_0$. Hence, for all $\varepsilon \in \{-1, 1\}$,

$$x \cap V(X_k - \varepsilon X_{k'}) = \begin{cases} x & \text{if } x \subseteq V(X_k - \varepsilon X_{k'}), \\ x \cap V(X_{\min(K)}) & \text{otherwise.} \end{cases}$$

Therefore, for $m = |\{I \mid I \in P \setminus \{P_0\}, |I| = 1\}|$,

$$\mathcal{D}_n^x \cong \{X_I \pm X_J \mid I, J \in P \setminus \{P_0\}, I \neq J\} \cup \{X_K \mid K \in P, |K| \geq 2\} \cong \mathcal{A}_{n-\text{rk}(x), m}. \quad \square$$

Although the arrangements $\mathcal{A}_{n,m}$ are not necessarily Coxeter arrangements, they are close enough: their Poincaré polynomials are determined by Poincaré polynomials of classical Coxeter arrangements. The next lemma applies the Deletion-Restriction Theorem [19, Thm. 2.56] to compute them.

Lemma 5.13. *For $n \in \mathbb{N}$ and $m \in [n]_0$, we have*

$$\pi_{\mathcal{A}_{n,m}}(Y) = (1 + (2n - m - 1)Y) \prod_{i=1}^{n-1} (1 + (2i - 1)Y).$$

Proof. We proceed by induction on m . If $m = 0$, then $\mathcal{A}_{n,m} \cong \mathcal{B}_n$ for all $n \in \mathbb{N}$, and the lemma holds; see, for instance, Example 5.4.

Now, let $m \in [n - 1]_0$. Since $\mathcal{A}_{n,m+1} = \mathcal{A}_{n,m} \setminus \{X_{n-m}\}$, let $x = V(X_{n-m})$. It follows that $\mathcal{A}_{n,m}^x \cong \mathcal{B}_{n-1}$. By the Deletion-Restriction Theorem,

$$\pi_{\mathcal{A}_{n,m+1}}(Y) = \pi_{\mathcal{A}_{n,m}}(Y) - Y \pi_{\mathcal{B}_{n-1}}(Y),$$

so the result follows by the induction hypothesis. \square

We now turn to the combinatorial specifications of Lemma 2.2 for the respective types.

5.4.1. Type A.

Lemma 5.14. *For all $n \geq 1$ and all $cDVR$ \mathfrak{o} with residue field cardinality q ,*

$$(5.10) \quad \zeta_{\mathcal{A}_n(\mathfrak{o})}^{\text{at}}((s_J)_{J \in \mathcal{P}(n+1;2)}) = \frac{1}{1 - q^{-n - \sum_{J \in \mathcal{P}(n+1;2)} s_J}} \sum_{\substack{P \in \Pi_{n+1} \\ |P| \geq 2}} (|P| - 1)_{-q^{-1}}! \\ \cdot \prod_{I \in P} q^{-(|I|-1) - \sum_{J \in \mathcal{P}(I;2)} s_J} \zeta_{\mathcal{A}_{|I|-1}(\mathfrak{o})}^{\text{at}}((s_J)_{J \in \mathcal{P}(I;2)}).$$

Proof. Set $\mathcal{A} = \mathcal{A}_n$. Then Lemma 2.2 (2.4) implies that

$$(5.11) \quad \zeta_{\mathcal{A}(\mathfrak{o})}^{\text{at}}((s_L)_{L \in \mathcal{A}}) = \frac{1}{1 - q^{-n - \sum_{L \in \mathcal{A}} s_L}} \sum_{x \in \mathcal{L}(\mathcal{A}) \setminus \{\hat{1}\}} q^{-\text{rk}(x) - \sum_{L \in \mathcal{A}_x} s_L} \\ \cdot \pi_{\mathcal{A}_x}(-q^{-1}) \zeta_{\mathcal{A}_x(\mathfrak{o})}^{\text{at}}((s_L)_{L \in \mathcal{A}_x}).$$

Let $P \in \Pi_{n+1}$ such that $|P| \geq 2$ and set $x = \rho_{\mathcal{A},n}(P) \in \tilde{\mathcal{L}}(\mathcal{A}) \setminus \{\hat{1}\}$, where $\rho_{\mathcal{A},n}$ is the isomorphism defined in (5.2). We show that the x -summand in (5.11) is the P -summand in (5.10). By Lemma 5.10, $\mathcal{A}^x \cong \mathbf{A}_{|P|-1}$ and $\mathcal{A}_x \cong \prod_{I \in P} \mathbf{A}_{|I|-1}$, so

$$\pi_{\mathcal{A}^x}(Y) = \prod_{k=1}^{|P|-1} (1+kY) = (|P|-1)_Y!,$$

and $\text{rk}(x) = n+1-|P|$. For $\mathbf{s} = ((s_J)_{J \in \mathcal{P}(I;2)})_{I \in P}$, since $\mathcal{A}_x \cong \prod_{I \in P} \mathbf{A}_{|I|-1}$,

$$\zeta_{\mathcal{A}_x(\mathfrak{o})}^{\text{at}}(\mathbf{s}) = \prod_{I \in P} \zeta_{\mathbf{A}_{|I|-1}(\mathfrak{o})}^{\text{at}}((s_J)_{J \in \mathcal{P}(I;2)}).$$

Hence, the lemma follows. \square

5.4.2. *Type B.* The proof of the recursive formula for type B is similar to the type-A case in Lemma 5.14, so we omit some details.

Lemma 5.15. *For all $n \geq 1$ and all cDVRs \mathfrak{o} with odd residue field cardinality q ,*

$$\begin{aligned} \zeta_{\mathbf{B}_n(\mathfrak{o})}^{\text{at}}((s_J)_{J \in \mathcal{P}_{\mathbf{B}}([n]_0;2)}) &= \frac{1}{1 - q^{-n - \sum_{J \in \mathcal{P}_{\mathbf{B}}([n]_0;2)} s_J}} \sum_{\substack{P \in \Pi_{\mathbf{B},n} \\ |P| \geq 2}} (|P|-1)_{-q^{-1}}! \\ &\cdot q^{-(|P_0|-1) - \sum_{J \in \mathcal{P}_{\mathbf{B}}(P_0;2)} s_J} \zeta_{\mathbf{B}_{|P_0|-1}(\mathfrak{o})}^{\text{at}}((s_J)_{J \in \mathcal{P}_{\mathbf{B}}(P_0;2)}) \\ &\cdot \prod_{I \in P \setminus \{P_0\}} q^{-(|I|-1) - \sum_{J \in \mathcal{P}(I;2)} s_J} \zeta_{\mathbf{A}_{|I|-1}(\mathfrak{o})}^{\text{at}}((s_J)_{J \in \mathcal{P}(I;2)}). \end{aligned}$$

Proof. Set $\mathcal{A} = \mathbf{B}_n$. Let $P \in \Pi_{\mathbf{B},n}$ such that $|P| \geq 2$, and set $x = \rho_{\mathbf{B},n}(P) \in \tilde{\mathcal{L}}(\mathcal{A}) \setminus \{\hat{1}\}$, where $\rho_{\mathbf{B},n}$ is the isomorphism defined in (5.3). Using Lemma 5.10, $\mathcal{A}_x \cong \mathbf{B}_{|P_0|-1} \times \prod_{I \in P \setminus \{P_0\}} \mathbf{A}_{|I|-1}$ and $\mathcal{A}^x \cong \mathbf{B}_{|P|-1}$. The rank of x is $n+1-|P|$, the Poincaré polynomial is $\pi_{\mathcal{A}^x}(Y) = (|P|-1)_Y!$. For $\mathbf{s} = ((s_J)_{J \in \mathcal{P}_{\mathbf{B}}(I;2)})_{I \in P}$,

$$\zeta_{\mathcal{A}_x(\mathfrak{o})}^{\text{at}}(\mathbf{s}) = \zeta_{\mathbf{B}_{|P_0|-1}(\mathfrak{o})}^{\text{at}}((s_J)_{J \in \mathcal{P}_{\mathbf{B}}(P_0;2)}) \prod_{I \in P \setminus \{P_0\}} \zeta_{\mathbf{A}_{|I|-1}(\mathfrak{o})}^{\text{at}}((s_J)_{J \in \mathcal{P}(I;2)}). \quad \square$$

5.4.3. *Type D.* Recall that, for a central arrangement \mathcal{A} , the sum in Lemma 2.2 (2.4) runs through $\mathcal{L}(\mathcal{A}) \setminus \{\hat{1}\}$. As seen in Lemma 5.12, there are two different cases of set partitions in $\Pi_{\mathbf{D},n} \cong \mathcal{L}(\mathbf{D}_n)$, so we split the sum in Lemma 2.2 (2.4) into two parts as follows: let

- $\zeta_{\mathbf{D}_n(\mathfrak{o}),0}^{\text{at}}(\mathbf{s})$ be the sum running through the $x \in \mathcal{L}(\mathbf{D}_n)$ such that x is not contained in $V(X_k)$ for all $k \in [n]$, and
- $\zeta_{\mathbf{D}_n(\mathfrak{o}),+}^{\text{at}}(\mathbf{s})$ be the sum running through the $x \in \mathcal{L}(\mathbf{D}_n)$ such that there exists $k \in [n]$ such that $x \subseteq V(X_k)$.

In other words, we split the sum for $\zeta_{\mathbf{D}_n(\mathfrak{o})}^{\text{at}}(\mathbf{s})$ based on whether $x \in \mathcal{L}(\mathbf{D}_n)$ is on a coordinate hyperplane or not. In terms of set partitions of type D, the summand $\zeta_{\mathbf{D}_n(\mathfrak{o}),0}^{\text{at}}(\mathbf{s})$ is the sum over set partitions with zero block equal to $\{0\}$, whereas $\zeta_{\mathbf{D}_n(\mathfrak{o}),+}^{\text{at}}(\mathbf{s})$ is the sum over set partitions with zero block not equal to $\{0\}$. Of course, $\zeta_{\mathbf{D}_n(\mathfrak{o})}^{\text{at}}(\mathbf{s}) = \zeta_{\mathbf{D}_n(\mathfrak{o}),0}^{\text{at}}(\mathbf{s}) + \zeta_{\mathbf{D}_n(\mathfrak{o}),+}^{\text{at}}(\mathbf{s})$.

For $P \in \Pi_{\mathbf{D},n}$, denote the number of singleton nonzero blocks of P by

$$(5.12) \quad \text{Sng}(P) := |\{I \mid I \in P \setminus \{P_0\}, |I| = 1\}|.$$

Lemma 5.16. *Let $n \geq 2$. For all cDVRs \mathfrak{o} with odd residue field cardinality q ,*

$$\zeta_{\mathbf{D}_n(\mathfrak{o}),0}^{\text{at}}(\mathbf{s}) = \frac{1}{1 - q^{-n - \sum_{J \in \mathcal{P}_{\mathbf{D}}([n]_0;2)} s_J}} \sum_{\substack{P \in \Pi_{\mathbf{D},n} \\ P_0 = \{0\}}} ((2|P| - \text{Sng}(P) - 3))_{-q^{-1}}$$

$$\begin{aligned}
& \cdot (|P| - 2)_{-q^{-1}}!! \prod_{I \in P \setminus \{P_0\}} q^{-(|I|-1) - \sum_{J \in \mathcal{P}(I;2)} s_J} \zeta_{\mathcal{A}_{|I|-1}(\mathfrak{o})}^{\text{at}}((s_J)_{J \in \mathcal{P}(I;2)}), \\
\zeta_{\mathcal{D}_n(\mathfrak{o}),+}^{\text{at}}(\mathfrak{s}) &= \frac{1}{1 - q^{-n - \sum_{J \in \mathcal{P}_D(\{n\}_0;2)} s_J}} \sum_{\substack{P \in \Pi_{\mathcal{D},n} \\ 3 \leq |P_0| \leq n}} (|P| - 1)_{-q^{-1}}!! \\
& \cdot q^{-(|P_0|-1) - \sum_{J \in \mathcal{P}_D(P_0;2)} s_J} \zeta_{\mathcal{D}_{|P_0|-1}(\mathfrak{o})}^{\text{at}}((s_J)_{J \in \mathcal{P}_D(P_0;2)}) \\
& \cdot \prod_{I \in P \setminus \{P_0\}} q^{-(|I|-1) - \sum_{J \in \mathcal{P}(I;2)} s_J} \zeta_{\mathcal{A}_{|I|-1}(\mathfrak{o})}^{\text{at}}((s_J)_{J \in \mathcal{P}(I;2)}).
\end{aligned}$$

Proof. Let $P \in \Pi_{\mathcal{D},n}$ such that $|P| \geq 2$, and set $x = \rho_{\mathcal{D},n}(P)$, where $\rho_{\mathcal{D},n}$ is the restriction of the isomorphism defined in (5.3). It follows that $(\mathcal{D}_n)_x \cong \mathcal{D}_{|P_0|-1} \times \prod_{I \in P \setminus \{P_0\}} \mathcal{A}_{|I|-1}$ and that $\text{rk}(x) = n - |P| + 1$.

Now we consider two cases based on the cardinality $|P_0|$. First, suppose $|P_0| = 1$, and recall from Definition 5.11 the arrangement $\mathcal{A}_{n,m}$. The proof of Lemma 5.12 implies that $\mathcal{D}_n^x \cong \mathcal{A}_{|P_0|-1, \text{Sng}(P)}$. By Lemma 5.13,

$$\begin{aligned}
\pi_{\mathcal{D}_n^x}(Y) &= (1 + (2|P| - \text{Sng}(P) - 3)Y) \prod_{i=1}^{|P|-2} (1 + (2i - 1)Y) \\
&= ((2|P| - \text{Sng}(P) - 3)_Y (|P| - 2)_Y)!.
\end{aligned}$$

Now suppose that $|P_0| > 1$. Recall that set partitions of type \mathcal{D} have zero blocks of size different from 2, so $3 \leq |P_0| \leq n$ in this case. Then $\mathcal{D}_n^x \cong \mathcal{B}_{|P_0|-1}$ by Lemma 5.12, so the lemma follows. \square

5.4.4. *Proof of Theorem 5.6.* Apply Lemmas 5.14, 5.15, or 5.16 recursively. We observe that because the summations in each of the lemmas require the set partition $P \in \Pi_{\mathcal{X},n}$ to have size at least 2, the final sum runs through all total partitions. At every instance in the recursion, $|P|$ is the number of children of some parent vertex in τ . In type \mathcal{D} , if $P \in \Pi_{\mathcal{D},n}$ such that $|P_0| = 1$, then, for some τ , the number $\text{Sng}(P) + 1$ is the number of children of $v_0^+(\tau)$ that are unbranched.

This completes the proof of Theorem 5.6 and thus establishes Theorem C. \square

6. COARSE FLAG HILBERT–POINCARÉ SERIES

Our main aim in this section is to prove Theorem D. In Section 6.1 we record some results pertaining to coarse flag Hilbert–Poincaré series of general (not necessarily Coxeter) hyperplane arrangements, including some of their special values, behavior at $Y = 0$, and Hadamard products. We give explicit formulae for Boolean and generic central arrangements in Section 6.2. (Many more can be found in Appendix A.) In Section 6.3 we prove Theorem 6.10, the special case of Theorem D for classical arrangements. In Section 6.4 we combine these results to prove Theorem D.

6.1. General properties of coarse flag Hilbert–Poincaré series. In this section, let \mathcal{A} be an arbitrary hyperplane arrangement, i.e. not necessarily defined over a field of characteristic zero. Recall the definition (1.13) of the numerator polynomial $\mathcal{N}_{\mathcal{A}}(Y, T)$ of the coarse flag Hilbert–Poincaré series $\text{cfHP}_{\mathcal{A}}(Y, T)$.

We start with a general observation. We denote by $\max(\Delta)$ the set of maximal-dimensional faces of a simplicial complex Δ .

Lemma 6.1. *The following hold:*

- (1) $\text{cfHP}_{\mathcal{A}}(Y, 0) = \pi_{\mathcal{A}}(Y)$,
- (2) $\mathcal{N}_{\mathcal{A}}(Y, 1) = |\max(\Delta(\tilde{\mathcal{L}}(\mathcal{A}))| (1 + Y)^{\text{rk}(\mathcal{A})}$.

Proof. Setting $m = \text{rk}(\mathcal{A}) - \delta_{\widehat{1} \in \mathcal{L}(\mathcal{A})}$, both follow from the equation

$$\mathcal{N}_{\mathcal{A}}(Y, T) = \sum_{F \in \Delta(\overline{\mathcal{L}}(\mathcal{A}))} \pi_F(Y) T^{|F|} (1 - T)^{m - |F|}. \quad \square$$

It is known that there exist inequivalent arrangements with the same Poincaré polynomial [19, Ex. 2.61]. All of our results and computations suggest that the following question may have a negative answer.

Question 6.2. Do there exist two inequivalent arrangements \mathcal{A} and \mathcal{B} such that $\text{cfHP}_{\mathcal{A}}(Y, T) = \text{cfHP}_{\mathcal{B}}(Y, T)$?

6.1.1. *Behavior at $Y = 0$.* Conjecture E is partly motivated by Proposition 1.9, which follows immediately from deep yet well-known results in poset topology.

Proof of Proposition 1.9. The first part of the statement follows from Lemma 6.1(1). For the second part, let $\Delta := \Delta(\mathcal{L}(\mathcal{A}))$. Let \mathbb{F} be a field and $\mathbb{F}[\Delta]$ be the Stanley–Reisner ring of Δ . Since $\pi_F(0) = 1$ for all $F \in \Delta$, we find that

$$\frac{1}{1 - T_0} \text{fHP}_{\mathcal{A}}(0, T) = \sum_{F \in \Delta} \prod_{x \in F} \frac{T_x}{1 - T_x} = \text{Hilb}(\mathbb{F}[\Delta], T),$$

the fine Hilbert series of $\mathbb{F}[\Delta]$; cf. [21, Chap. 10.6]. As $\mathcal{L}(\mathcal{A})$ is a geometric semilattice, Δ is Cohen–Macaulay over \mathbb{F} ; see, for instance, [35]. By [26, Cor. 3.2], there exists $h_k \in \mathbb{N}_0$, for $k \in [\text{rk}(\mathcal{A})]_0$, such that

$$\text{Hilb}(\mathbb{F}[\Delta], T) = \frac{\sum_{k=0}^{\text{rk}(\mathcal{A})} h_k T^k}{(1 - T)^{\text{rk}(\mathcal{A})+1}} = \frac{1}{1 - T} \text{cfHP}_{\mathcal{A}}(0, T). \quad \square$$

6.1.2. *Hadamard products.* In this section, we consider the effect of taking direct products of hyperplane arrangements on flag Hilbert–Poincaré series. This turns out to be described by Hadamard products of a moderate variant of $\text{cfHP}_{\mathcal{A}}(Y, T)$. Set

$$\widehat{\text{cfHP}}_{\mathcal{A}}(Y, T) := \sum_{F \in \Delta(\mathcal{L}(\mathcal{A}))} \pi_F(Y) \left(\frac{T}{1 - T} \right)^{|F|} = \frac{\text{cfHP}_{\mathcal{A}}(Y, T)}{1 - T}.$$

Let us expand $\widehat{\text{cfHP}}_{\mathcal{A}}(Y, T)$ as a generating function. First define $\alpha_0(\mathcal{A}; Y) := \pi_{\mathcal{A}}(Y)$ and, for $k \geq 1$,

$$\alpha_k(\mathcal{A}; Y) := \sum_{\substack{F \in \Delta(\mathcal{L}(\mathcal{A})) \\ 1 \leq |F| \leq k}} \binom{k-1}{|F|-1} \pi_F(Y).$$

Since $(T/(1 - T))^f = \sum_{k \geq 0} \binom{k+f-1}{f-1} T^{f+k}$ for all $f \in \mathbb{N}$, it follows that

$$(6.1) \quad \widehat{\text{cfHP}}_{\mathcal{A}}(Y, T) = \sum_{k \geq 0} \alpha_k(\mathcal{A}; Y) T^k.$$

Recall that, given sequences $(\varphi_k), (\psi_k) \in \mathbb{Q}^{\mathbb{N}_0}$, the *Hadamard product* of the generating functions $f(T) = \sum_{k \geq 0} \varphi_k T^k$ and $g(T) = \sum_{k \geq 0} \psi_k T^k$ along T is

$$f(T) \star_T g(T) = \sum_{k \geq 0} \varphi_k \psi_k T^k.$$

Proposition 6.3. *Given hyperplane arrangements \mathcal{A}_1 and \mathcal{A}_2 , we have*

$$\widehat{\text{cfHP}}_{\mathcal{A}_1 \times \mathcal{A}_2}(Y, T) = \widehat{\text{cfHP}}_{\mathcal{A}_1}(Y, T) \star_T \widehat{\text{cfHP}}_{\mathcal{A}_2}(Y, T).$$

Before proving Proposition 6.3, we prove a lemma using basic facts about the *Delannoy numbers* $D(m, n)$. These count the number of lattice paths from $(0, 0)$ to $(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$ using steps in $\{(1, 0), (1, 1), (0, 1)\}$; cf., e.g., [9, p. 81]. Let $D(m, n, \ell) = \binom{\ell-1}{m} \binom{m}{\ell-n-1}$ be the number of such lattice paths traversing exactly ℓ vertices. Hence

$$D(m, n) = \sum_{\ell=1}^{m+n+1} D(m, n, \ell) = \sum_{\ell=1}^{m+n+1} \binom{\ell-1}{m} \binom{m}{\ell-n-1}.$$

We shall use the following notation in the sequel. If F is a weakly increasing flag, we denote by $F^<$ the maximal strictly increasing sub-flag of F .

Lemma 6.4. *Given hyperplane arrangements \mathcal{A}_1 and \mathcal{A}_2 , we have*

$$\widehat{\text{cfHP}}_{\mathcal{A}_1 \times \mathcal{A}_2}(0, T) = \widehat{\text{cfHP}}_{\mathcal{A}_1}(0, T) \star_T \widehat{\text{cfHP}}_{\mathcal{A}_2}(0, T).$$

Proof. By (6.1), it suffices to show that, for all $k \geq 0$,

$$(6.2) \quad \alpha_k(\mathcal{A}_1 \times \mathcal{A}_2; 0) = \alpha_k(\mathcal{A}_1; 0) \alpha_k(\mathcal{A}_2; 0).$$

This is clear for $k = 0$, so we assume $k \geq 1$ is fixed. Note that $\mathcal{L}(\mathcal{A}_1 \times \mathcal{A}_2) \cong \mathcal{L}(\mathcal{A}_1) \times \mathcal{L}(\mathcal{A}_2)$. The latter is partially ordered as follows: $(x_1, x_2) \leq (y_1, y_2)$ if $x_1 \leq y_1$ and $x_2 \leq y_2$; see [19, Prop. 2.14]. For $i \in \{1, 2\}$, let $\varphi_i : \mathcal{L}(\mathcal{A}_1 \times \mathcal{A}_2) \rightarrow \mathcal{L}(\mathcal{A}_i)$ be the respective projection.

For $i \in \{1, 2\}$, fix nonempty $G_i \in \Delta(\mathcal{L}(\mathcal{A}_i))$, and set

$$\mathcal{F}_k(G_1, G_2) = \{(x_1 < \cdots < x_\ell) \in \Delta(\mathcal{L}(\mathcal{A}_1 \times \mathcal{A}_2)) \mid \\ \ell \in [k], \forall i \in \{1, 2\} : G_i = (\varphi_i(x_1) \leq \cdots \leq \varphi_i(x_\ell))^<\}.$$

The set of flags $\mathcal{F}_k(G_1, G_2)$ is in bijection with the set of lattice paths from $(0, 0)$ to $(|G_1| - 1, |G_2| - 1)$ with steps in $\{(1, 0), (1, 1), (0, 1)\}$, traversing no more than k lattice points. For $\ell \in \mathbb{N}$ we define, in addition,

$$\mathcal{F}(G_1, G_2, \ell) = \{(x_1 < \cdots < x_\ell) \in \Delta(\mathcal{L}(\mathcal{A}_1 \times \mathcal{A}_2)) \mid \\ \forall i \in \{1, 2\} : G_i = (\varphi_i(x_1) \leq \cdots \leq \varphi_i(x_\ell))^<\}.$$

Flags in $\mathcal{F}_k(G_1, G_2)$ have lengths between 1 and k , whereas flags in $\mathcal{F}(G_1, G_2, \ell)$ all have length ℓ . The latter set is in bijection with the set of Delannoy paths from $(0, 0)$ to $(|G_1| - 1, |G_2| - 1)$ traversing ℓ vertices. Considering the partition

$$\mathcal{F}_k(G_1, G_2) = \bigcup_{\ell=1}^k \mathcal{F}(G_1, G_2, \ell).$$

and setting $m = |G_1| - 1$ and $n = |G_2| - 1$, we obtain

$$(6.3) \quad \sum_{F \in \mathcal{F}_k(G_1, G_2)} \binom{k-1}{|F|-1} = \sum_{\ell=1}^k \binom{k-1}{\ell-1} D(m, n, \ell) \\ = \binom{k-1}{m} \binom{k-1}{n} = \binom{k-1}{|G_1|-1} \binom{k-1}{|G_2|-1}.$$

Here, the penultimate equality is obtained by counting the subsets of the form $A \times B$, for $A, B \subseteq [k-1]$ with $|A| = m$ and $|B| = n$, setting $|A \cup B| = \ell - 1$. Since (6.3) holds for all nonempty $G_1 \in \Delta(\mathcal{L}(\mathcal{A}_1))$ and $G_2 \in \Delta(\mathcal{L}(\mathcal{A}_2))$, equation (6.2) holds. \square

Remark 6.5. The second equality in (6.3) seems reminiscent of, but distinct from Saalschütz's identity; see [13, (5.28)].

Question 6.6. Can the Hadamard factorization in Lemma 6.4 be deduced directly from the interpretation $\widehat{\text{cfHP}}_{\mathcal{A}}(0, T) = \text{Hilb}(\mathbb{F}[\Delta(\mathcal{L}(\mathcal{A}))], T)$?

Proof of Proposition 6.3. Fix an isomorphism $\varphi : \mathcal{L}(\mathcal{A}_1 \times \mathcal{A}_2) \rightarrow \mathcal{L}(\mathcal{A}_1) \times \mathcal{L}(\mathcal{A}_2)$. For $x \in \mathcal{L}(\mathcal{A}_1 \times \mathcal{A}_2)$, write $\varphi_i(x)$ for $\varphi(x) = (\varphi_1(x), \varphi_2(x))$, and for $F = (x_1 < \cdots < x_k) \in \Delta(\mathcal{L}(\mathcal{A}_1 \times \mathcal{A}_2))$, define $G_i = (\varphi_i(x_1) \leq \cdots \leq \varphi_i(x_k))$ for each $i \in \{1, 2\}$. Note $\pi_{G_i}(Y) = \pi_{G_i^c}(Y)$. Since $\pi_{\mathcal{A} \times \mathcal{A}'}(Y) = \pi_{\mathcal{A}}(Y)\pi_{\mathcal{A}'}(Y)$ for hyperplane arrangements \mathcal{A} and \mathcal{A}' (cf. [19, Lem. 2.50]), it follows that $\pi_F(Y) = \pi_{G_1^c}(Y)\pi_{G_2^c}(Y)$. The proposition follows from Lemma 6.4. \square

6.2. Families of examples. We record formulae of coarse flag Hilbert–Poincaré series for some infinite families of examples. Appendix A contains many more.

6.2.1. Boolean arrangements. Recall the definition (1.14) of the n th Eulerian polynomial $E_n(T)$ and the definition (4.2) of the weak order zeta function $\mathcal{I}_n^{\text{WO}}(\mathbf{T})$. Specializing $T_I = T$ for each nonempty $I \subseteq [n]$ in (4.2) yields

$$\mathcal{I}_n^{\text{WO}}(T) = \frac{E_n(T)}{(1-T)^n}.$$

The following result is thus an immediate consequence of Proposition 4.5.

Proposition 6.7. *For $n \geq 1$,*

$$\text{cfHP}_{\mathcal{A}_1^n}(Y, T) = (1+Y)^n \frac{E_n(T)}{(1-T)^n}.$$

6.2.2. Generic central arrangements. Proposition 4.7 implies the following result.

Proposition 6.8. *Let $m, n \in \mathbb{N}$ with $n \leq m$. Then*

$$\begin{aligned} \text{cfHP}_{\mathcal{U}_{n,m}}(Y, T) &= \frac{1+Y}{1-T} \sum_{k=0}^{n-1} \binom{m-1}{k} Y^k \\ &\quad + \sum_{\ell=1}^{n-1} \sum_{k=0}^{n-\ell-1} \binom{m-\ell-1}{k} \binom{m}{\ell} \frac{(1+Y)^{\ell+1} Y^k T \cdot E_\ell(T)}{(1-T)^{\ell+1}}. \end{aligned}$$

For $m \geq n = 2$, Proposition 6.8 recovers a consequence of Proposition 4.2, viz.

$$\mathcal{N}_{\mathcal{U}_{2,m}}(Y, T) = (1+Y)(1+(m-1)(Y+T)+YT).$$

For $m \geq n = 3$, Proposition 6.8 states that

$$\begin{aligned} \mathcal{N}_{\mathcal{U}_{3,m}}(Y, T) &= (1+Y) \left\{ (1+Y^2T^2) + (m-1)Y(1+T^2) + 2(m-1)^2YT \right. \\ &\quad \left. + \left(\binom{m+1}{2} - 2 \right) T(1+Y^2) + \left(\frac{m(m-3)}{2} + 1 \right) (Y^2+T^2) \right\}. \end{aligned}$$

With Lemma 6.1, we obtain $\pi_{\mathcal{U}_{3,m}}(1) = \mathcal{N}_{\mathcal{U}_{3,m}}(1, 0) = m^2 - m + 2$ and hence

$$(6.4) \quad \frac{\mathcal{N}_{\mathcal{U}_{3,m}}(1, T)}{\pi_{\mathcal{U}_{3,m}}(1)} = 1 + \left(6 - \frac{16}{m^2 - m + 2} \right) T + T^2.$$

Thus the polynomial in (6.4) is in $\mathbb{Z}[T]$ if and only if $m = 3$. We note that $m = 3$ is the unique value for which $\mathcal{U}_{3,m}$ is a Coxeter arrangement.

We have found only three pairs $(n, m) \in [1000]^2$ with $m > n > 3$ with the property that $\mathcal{N}_{\mathcal{U}_{n,m}}(1, T)/\pi_{\mathcal{U}_{n,m}}(1) \in \mathbb{Z}[T]$, namely those in $\{(4, 5), (4, 7), (4, 8)\}$; see Section A.5 for explicit formulae. The associated normalized integral polynomials at $Y = 1$ are, respectively,

$$1 + 15T + 15T^2 + T^3, \quad 1 + 19T + 19T^2 + T^3, \quad 1 + 20T + 20T^2 + T^3.$$

From this perspective, it seems rare that $\mathcal{N}_{\mathcal{A}}(1, T)/\pi_{\mathcal{A}}(1) \in \mathbb{Z}[T]$; indeed, of all the non-Coxeter examples computed in Appendix A, only $\mathcal{U}_{4,5}$, $\mathcal{U}_{4,7}$, and $\mathcal{U}_{4,8}$ satisfy this property.

Question 6.9.

- (1) For which $m > n$ does the following hold:

$$\mathcal{N}_{\mathcal{U}_{n,m}}(1, T) / \pi_{\mathcal{U}_{n,m}}(1) \in \mathbb{Z}[T]?$$

- (2) Is there an infinite set \mathcal{F} of non-Coxeter arrangements such that

$$\mathcal{N}_{\mathcal{A}}(1, T) / \pi_{\mathcal{A}}(1) \in \mathbb{Z}[T]$$

for all $\mathcal{A} \in \mathcal{F}$?

- (3) Is there an infinite set \mathcal{G} of arrangements with the property that

$$\mathcal{N}_{\mathcal{A}}(1, T) / \pi_{\mathcal{A}}(1) \neq E_m(T)$$

such that for all $\mathcal{A} \in \mathcal{G}$ and $m \in \mathbb{N}$?

We remark that the non-Coxeter restrictions of D_n are a good candidate family for answering Question 6.9 (2) positively; see Definition 5.11 and Section A.2.

6.3. Classical Coxeter arrangements. Here we prove Theorem D for classical Coxeter arrangements. This will be the key step in the proof of the general case given in Section 6.4. Recall that $S(n, k)$ are the Stirling numbers of the second kind. We denote the set of flags of $\bar{\mathcal{L}}(\mathcal{A})$ with length $k \in \mathbb{N}_0$ by $\Delta_k(\bar{\mathcal{L}}(\mathcal{A}))$.

Theorem 6.10 (Theorem D for classical types). *For $1 \leq k \leq n$ and $X \in \{A, B, D\}$,*

$$\sum_{F \in \Delta_{k-1}(\bar{\mathcal{L}}(X_n))} \frac{\pi_F(1)}{\pi_{X_n}(1)} = k! S(n, k).$$

The proof of Theorem 6.10 will occupy most of the rest of this section and be completed in Section 6.3.3.

6.3.1. Rooted and plane trees. To prove Theorem 6.10, we enumerate sets of rooted trees using various maps which we describe in turn. We build on terminology from [29, Appendix], specifically for (plane) rooted trees, and from Sections 5.2 and 5.3. In particular we transfer terminology for X_n -labeled rooted trees introduced in Definition 5.1 and ensuing identifications to plane trees.

The k th generation of a rooted tree τ comprises all the vertices of distance k from the root; i.e. one traverses k distinct edges from the root to such a vertex. The k th generation of τ is *nontrivial* if there exists a vertex, of distance k from the root, with at least two children.

Definition 6.11. Let $n, k \in \mathbb{N}$ and $X \in \{A, B, D\}$.

- Let $\text{PT}_{n,k}$ be the set of (unlabeled) plane trees with n leaves and k generations, all of which are nontrivial; without loss of generality, all leaves are in the k th generation.
- Let $\text{LPT}_{X,n,k}$ resp. $\text{LRT}_{X,n,k}$ be the set of X_n -labeled plane resp. rooted trees with k generations, all of which are nontrivial; without loss of generality, all leaves are in the k th generation.

We remark that trees of $\text{PT}_{n,k}$ are said to be of *length* k in [29, Appendix], and the trees in $\text{PT}_{n,k}$ have one less leaf than trees in $\text{LPT}_{X,n,k}$.

The following lemma, which gives a simple formula for the cardinality of $\text{PT}_{n+1,k}$ in terms of Stirling numbers of the second kind, may be well known. It is implicit in Cayley's work [7] from 1859. Having failed to locate it in the modern literature, we include its proof.

Lemma 6.12. *For $1 \leq k \leq n$, $|\text{PT}_{n+1,k}| = k! S(n, k)$.*

Proof. The quantity $k! S(n, k)$ is the number of words of length n on the alphabet $[k-1]_0$, where each integer is used at least once. It thus suffices to identify such words with trees in $\text{PT}_{n+1, k}$.

Let $\tau \in \text{PT}_{n+1, k}$ and label the leaves of τ from 1 to $n+1$ from left to right. We obtain a word $w(\tau)$ of length n on the alphabet $[k-1]_0$ whose i th character is the generation number of the common ancestor of leaves labeled i and $i+1$. If the leaves labeled i and $i+1$ are siblings, then the i th character of $w(\tau)$ is $k-1$.

We claim that each $j \in [k-1]_0$ appears in $w(\tau)$. Indeed, since τ has k generations there exists a vertex in the j th generation with at least two consecutive children, say u_1 followed by u_2 . Then the rightmost leaf that is a descendant of u_1 is the left neighbor of the leftmost leaf that is a descendant of u_2 . Hence, j appears in $w(\tau)$.

Reversing these steps for a word of length n on the alphabet $[k-1]_0$ where each integer is used at least once determines a unique tree $\tau \in \text{PT}_{n+1, k}$. \square

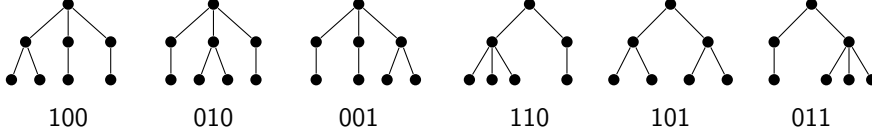


FIGURE 6.1. Illustration of the proof of Lemma 6.12 for $n = 3$ and $n = 2$. The six elements of $\text{PT}_{4,2}$ match with the words in $\{0, 1\}$ of length 3 which feature both characters.

To define the following maps, let $1 \leq k \leq n$ and $X \in \{A, B, D\}$. Let

$$\Lambda_{X,n,k} : \text{LPT}_{X,n,k} \rightarrow \text{PT}_{n+1,k}$$

be the map that forgets the labels. For types $X \in \{A, B\}$, these are surjections; the map $\Lambda_{D,n,k}$, however, need not be surjective. Observe that, for $\tau' \in \text{PT}_{n+1,k}$,

$$\left| \Lambda_{A,n,k}^{-1}(\tau') \right| = (n+1)!, \quad \left| \Lambda_{B,n,k}^{-1}(\tau') \right| = 2^n (n+1)!.$$

Additionally, let

$$\Omega_{X,n,k} : \text{LPT}_{X,n,k} \twoheadrightarrow \text{LRT}_{X,n,k}$$

be the surjection that forgets the ordering.

The isomorphism $\rho_{A,n} : \Pi_{A,n} \rightarrow \mathcal{L}(A_n)$ in (5.2) induces the isomorphism

$$\Phi_{A,n,k} : \text{LRT}_{A,n,k} \rightarrow \Delta_{k-1}(\overline{\mathcal{L}}(A_n)).$$

As explained in Section 5.2.2, every generation of a tree $\tau \in \text{LRT}_{B,n,k}$ determines a partition in $\Pi_{B,n}$. In this way τ determines a chain of length $k-1$ in $\overline{\mathcal{L}}(B_n)$, and every such chain arises in this way. Let

$$\Phi_{B,n,k} : \text{LRT}_{B,n,k} \twoheadrightarrow \Delta_{k-1}(\overline{\mathcal{L}}(B_n))$$

denote this surjection, and define

$$\Phi_{D,n,k} : \text{LRT}_{D,n,k} \twoheadrightarrow \Delta_{k-1}(\overline{\mathcal{L}}(D_n))$$

as the restriction of $\Phi_{B,n,k}$. Let

$$\beta_{n,k} : \text{LRT}_{B,n,k} \rightarrow \text{LRT}_{A,n,k}$$

be the function that forgets the bars and replaces the label 0 with the label $n+1$. (There are similarly defined maps on the order complex and sets of labeled plane trees, for which we will have no use.) Since $\Pi_{D,n} \subset \Pi_{B,n}$, there are natural embeddings defined on the order complex and labeled plane resp. rooted trees. Figure 6.2 summarizes these maps.

Recall Definition 5.3 for the standard form of $\tau \in \text{LRT}_{X,n,k} \subset \text{LRT}_{X,n}$.

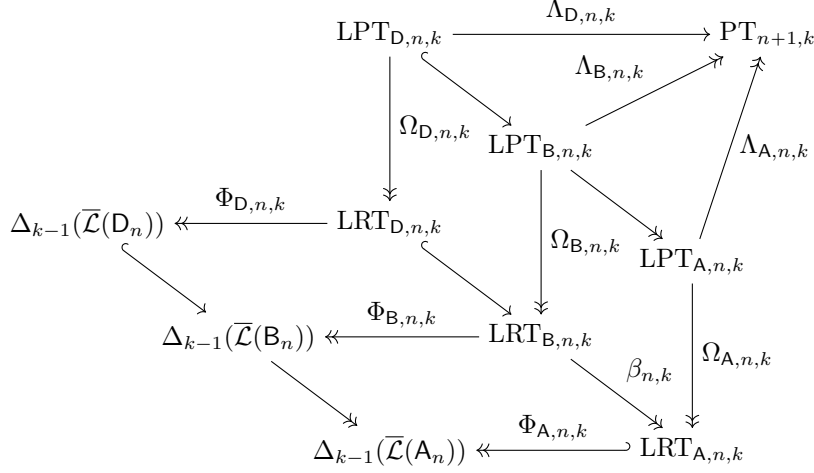


FIGURE 6.2. A commutative diagram showing all of the maps defined above with $1 \leq k \leq n$.

Lemma 6.13. *Let $X \in \{A, B, D\}$ and $1 \leq k \leq n$. For all $F \in \Delta_{k-1}(\overline{\mathcal{L}}(X_n))$, the fiber $\Phi_{X,n,k}^{-1}(F)$ contains a unique tree in standard form and for all $\tau \in \Phi_{X,n,k}^{-1}(F)$,*

$$\left| \Phi_{X,n,k}^{-1}(F) \right| = \prod_{u \in V(\tau, 0)} 2^{c(\tau, u) - 1}.$$

Proof. Every tree in $LRT_{A,n,k}$ is standard, and $\Phi_{A,n,k}$ is an isomorphism. It suffices, then, to prove this for type B since $\Phi_{D,n,k}$ is the restriction of $\Phi_{B,n,k}$. If $k = 1$, this is immediate, so we suppose $k \geq 2$ and $F = (x_1 < \dots < x_{k-1})$. For all $i \in [k-1]$, let $P_i = \rho_{B,n}(x_i) \in \Pi_{B,n}$, where $\rho_{B,n}$ is as in Section 5.1, and set $P_k = \hat{1}$.

Let $\tau \in \Phi_{B,n,k}^{-1}(F)$ and $u \in V(\tau, 0)$, where u is in the i th generation of τ , for $i \in [k-1]_0$. The vertices in the i th generation of τ are in one-to-one correspondence with the blocks of P_{k-i} . In particular, $\lambda(\tau, u) = P_{k-i,0}$, the zero block of P_{k-i} . For each child u' of u not contained in $V(\tau, 0)$, let $\alpha' \in [n]_0$ be the minimal integer label in $\text{DLL}(\tau, u')$; cf. Definition 5.2. It follows that α' is not barred in any P_j since α' is always either the minimal integer or contained in the zero block. Thus, if τ is labeled such that α' is barred, then there exists $\tau' \in LRT_{B,n,k}$ such that α' is not barred and $\Phi_{B,n,k}(\tau') = \Phi_{B,n,k}(\tau)$. This is the only way two trees in $LRT_{B,n,k}$ may have the same image. Thus there exists $\tau_{\text{st}} \in \Phi_{B,n,k}^{-1}(F)$ such that, for all $u \in V(\tau_{\text{st}}, 0)$ and for all children u' of u not in $V(\tau_{\text{st}}, 0)$, the minimal label α' of $\text{DLL}(\tau_{\text{st}}, u')$ is not barred; by Definition 5.3, the tree τ_{st} is in standard form, and the lemma follows. \square

Recall the definition (5.7) of $\pi_{X,\tau}(Y)$ for $\tau \in LRT_{X,n}$ in Section 5.3.

Lemma 6.14. *Let $X \in \{A, B, D\}$ and $1 \leq k \leq n$. If $F \in \Delta_{k-1}(\overline{\mathcal{L}}(X_n))$ then, for all $\tau \in \Phi_{X,n,k}^{-1}(F)$,*

$$\pi_F(Y) = \pi_{X,\tau}(Y).$$

Proof. First we show that, for $\tau, \tau' \in \Phi_{X,n,k}^{-1}(F)$,

$$(6.5) \quad \pi_{X,\tau}(Y) = \pi_{X,\tau'}(Y).$$

It suffices to prove (6.5) for $X = B$. From the proof of Lemma 6.13 we obtain $\beta_{n,k}(\tau) = \beta_{n,k}(\tau')$. Since $\pi_{X,\tau}(Y)$ depends only on the underlying (unlabeled) rooted tree and the leaf label 0, equation (6.5) holds.

Each generation in τ determines a set partition in $\Pi_{\mathbf{X},n}$; see Section 5.2.2 for details. For $\ell \in [k-1]_0$, let $P, Q \in \Pi_{\mathbf{X},n}$ be the set partitions associated with generations ℓ and $\ell+1$ respectively. Via $\rho_{\mathbf{X},n}$ defined in (5.2) or in (5.3), let $y = \rho_{\mathbf{X},n}(P)$ and $x = \rho_{\mathbf{X},n}(Q)$. The arrangement $(\mathbf{X}_n)_y$ is a product of classical Coxeter arrangements; the number of (possibly trivial) factors is equal the number of vertices in generation ℓ or, equivalently, $|P|$. Thus, $(\mathbf{X}_n)_y^x$ is a product of $|P|$ possibly trivial arrangements by Lemmas 5.10 and 5.12. A vertex u in generation ℓ corresponds to a trivial factor of $(\mathbf{X}_n)_y^x$ if and only if $\mathbf{c}(\tau, u) = 1$.

Let Γ be the induced subgraph of τ comprising vertices from generations ℓ and $\ell+1$, so Γ is a forest of rooted trees of length one. Up to a (re-)labeling of the vertices from generation $\ell+1$ (i.e. leaves of Γ) similar to the relabeling of polynomials in the proofs of Lemmas 5.10 and 5.12, it follows that Γ is a disjoint union of total partitions and rooted trees with exactly two vertices, the latter corresponding to the trivial factors. Since Poincaré polynomials are multiplicative over factors [19, Lemma 2.50], $\pi_{(\mathbf{X}_n)_y^x}(Y)$ is the product of the Poincaré polynomials of the disjoint total partitions of Γ . As a consequence of Theorem 5.6, the Poincaré polynomial of $\tau' \in \text{TP}_{\mathbf{X},n}$ is $\pi_{\mathbf{X},\tau'}(Y)$, which proves the lemma. \square

Recall the counting functions \mathbf{c} and \mathbf{u} defined in (5.5) and (5.6), respectively.

Lemma 6.15. *Let $\mathbf{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{D}\}$ and $1 \leq k \leq n$; if $\mathbf{X} = \mathbf{D}$, then we also assume that $n \geq 2$. If $F \in \Delta_{k-1}(\mathcal{L}(\mathbf{X}_n))$ and $\tau \in \Phi_{\mathbf{X},n,k}^{-1}(F)$, then*

$$\pi_F(1) = \left| \Phi_{\mathbf{X},n,k}^{-1}(F) \right| \left| \Omega_{\mathbf{X},n,k}^{-1}(\tau) \right| C_{\mathbf{X}}(\tau),$$

where $C_{\mathbf{A}}(\tau) = 1$ and

$$C_{\mathbf{X}}(\tau) = \begin{cases} \prod_{u \in V(\tau,0)} \mathbf{c}(\tau, u)^{-1} & \text{if } \mathbf{X} = \mathbf{B}, \\ \frac{2\mathbf{c}(\tau, v_0^+(\tau)) - \mathbf{u}(\tau, v_0^+(\tau)) - 1}{2(\mathbf{c}(\tau, v_0^+(\tau)) - 1)} \prod_{u \in V(\tau,0)} \mathbf{c}(\tau, u)^{-1} & \text{if } \mathbf{X} = \mathbf{D}. \end{cases}$$

Proof. By Lemma 6.14 it suffices to compute $\pi_{\mathbf{X},\tau}(1)$. We obtain these numbers by setting $Y = 1$ in (5.7), noting that

$$((n))_1! = (n+1)!, \quad \text{whereas} \quad ((n))_1!! = 2^n n! =: (2n)!!,$$

whence

$$\begin{aligned} \pi_{\mathbf{A},\tau}(1) &= \prod_{u \in P(\tau)} \mathbf{c}(\tau, u)!, \\ \pi_{\mathbf{B},\tau}(1) &= \left(\prod_{u \in V(\tau,0)} (2\mathbf{c}(\tau, u) - 2)!! \right) \left(\prod_{u \in P(\tau) \setminus V(\tau,0)} \mathbf{c}(\tau, u)! \right), \\ \pi_{\mathbf{D},\tau}(1) &= \frac{(2\mathbf{c}(\tau, v_0^+(\tau)) - \mathbf{u}(\tau, v_0^+(\tau)) - 1)}{2(\mathbf{c}(\tau, v_0^+(\tau)) - 1)} \pi_{\mathbf{B},\tau}(1). \end{aligned}$$

For all types \mathbf{X} , we have

$$\left| \Omega_{\mathbf{X},n,k}^{-1}(\tau) \right| = \prod_{u \in V(\tau)} \mathbf{c}(\tau, u)!,$$

and by Lemma 6.13,

$$\left| \Phi_{\mathbf{X},n,k}^{-1}(F) \right| = \prod_{u \in V(\tau,0)} 2^{\mathbf{c}(\tau, u) - 1}. \quad \square$$

Before embarking on a proof of Theorem 6.10, we need two more lemmas to round off the computations. Both are variations of the same counting arguments concerning the sums of probabilities modeled by decision trees.

It will be useful to discard most of the labels and work only with the set of “0-labeled plane trees,” viz. trees with one leaf labeled by 0 and all other leaves unlabeled. All of our counting functions are unaffected by this simplification.

Definition 6.16. Let $X \in \{\mathbf{B}, \mathbf{D}\}$ and $1 \leq k \leq n$.

- (1) Let $\text{LPT}_{X,n,k}^0$ be the set obtained from $\text{LPT}_{X,n,k}$ by removing all labels except for the label 0.
- (2) For $\tau \in \text{LPT}_{X,n,k}^0$, let $\text{Prob}(\tau) := \prod_{u \in V(\tau,0)} c(\tau, u)^{-1}$.

Lemma 6.17. Let $1 \leq k \leq n$. Then

$$\sum_{\tau \in \text{LPT}_{\mathbf{B},n,k}^0} \prod_{u \in V(\tau,0)} c(\tau, u)^{-1} = \frac{|\text{LPT}_{\mathbf{B},n,k}|}{n+1}.$$

Proof. It suffices to prove

$$(6.6) \quad \sum_{\tau \in \text{LPT}_{\mathbf{B},n,k}^0} \text{Prob}(\tau) = \frac{|\text{LPT}_{\mathbf{B},n,k}^0|}{n+1} = |\text{PT}_{n+1,k}|.$$

Let $\tau \in \text{PT}_{n+1,k}$, and fix any ordering of its leaves. For $i \in [n+1]$, let $\tau_i \in \text{LPT}_{\mathbf{B},n,k}^0$ be the plane tree obtained from labeling the i th leaf of τ by 0. Viewing τ as a decision tree, where every parent vertex assigns uniform probability to its children, we see that $\sum_{i=1}^{n+1} \text{Prob}(\tau_i) = 1$. This implies (6.6), which proves the lemma. \square

6.3.2. *Grafting plane trees.* Lemma 6.19 below is a type-D analog of Lemma 6.17. In order to prove it, we define a relation $\mathcal{G}_{n,k}$ (cf. Definition 6.18) on the set

$$\mathbf{T} = \text{LPT}_{\mathbf{B},n,k}^0 \times \text{LPT}_{\mathbf{D},n,k}^0.$$

This subset may be thought of as the set of pairs $(\sigma, \tau) \in \mathbf{T}$ with the property that τ is obtained from σ via an operation we call *grafting*, which we now explain.

Assume that $n \geq 2$, and let $\sigma \in \text{LPT}_{\mathbf{B},n,k}^0$. We call the subtree induced by a vertex $u \in V(\sigma)$ and all of its descendants the *u-branch* of σ . A *branch* of σ is a *u-branch* for some $u \in V(\sigma)$. We write $v_0^+ := v_0^+(\sigma)$ for the first ancestor of v_0 with at least two children. Suppose that v_0^+ is in generation g and write v_0^{++} for the first ancestor of v_0^+ with at least two children; see Figure 6.3 for an example.

Suppose that $\sigma \in \text{LPT}_{\mathbf{B},n,k}^0 \setminus \text{LPT}_{\mathbf{D},n,k}^0$. In this case it follows from the definitions that v_0^+ has exactly two children, both of which are unbranched; see Definitions 5.1 (3) and 6.11. Let u be the unique child of v_0^+ not contained in $V(\sigma, 0)$, and let B be the *u-branch* of σ . Note that B may be “on the right” or “on the left” of v_0^+ . We now remove the edge connecting v_0^+ and B and connect (or “graft”) B as a new branch to exactly one of $c(\sigma, v_0^{++}) - 1$ judiciously chosen “contemporaries” of v_0^+ (viz. vertices of σ of the same generation g). More precisely, let C be one of the $c(\sigma, v_0^{++}) - 1$ branches of σ whose root is a child of v_0^{++} and does not contain v_0^+ . We connect B onto the rightmost (resp. leftmost) branch contained in C at generation g if B is on the right (resp. left) of v_0 . We observe that each of the resulting $c(\sigma, v_0^{++}) - 1$ trees are elements of $\text{LPT}_{\mathbf{D},n,k}^0$ and call them the trees obtained from σ via *grafting*. Figure 6.3 illustrates an example with B on the right of v_0 .

If $\sigma \in \text{LPT}_{\mathbf{D},n,k}^0$, then, by definition, $\tau = \sigma$ is the only tree obtained from σ via grafting.

Definition 6.18. For $1 \leq k \leq n$ and $2 \leq n$, we set

$$\mathcal{G}_{n,k} = \{(\sigma, \tau) \in \mathbf{T} \mid \tau \text{ may be obtained from } \sigma \text{ by grafting}\}$$

and write $\sigma \sim \tau$ if $(\sigma, \tau) \in \mathcal{G}_{n,k}$.

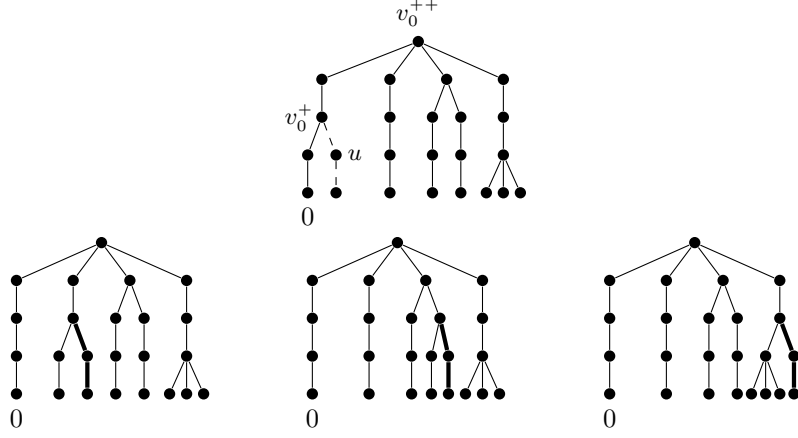


FIGURE 6.3. Via grafting, the top tree $\sigma \in \text{LPT}_{\mathbb{B},7,4}^0 \setminus \text{LPT}_{\mathbb{B},7,4}^0$ gives rise to the $c(\sigma, v_0^{++}) - 1 = 3$ bottom trees τ with $(\sigma, \tau) \in \mathcal{G}_{7,4}$. One removes the edge connecting v_0^+ and u and grafts the u -branch onto each of the three other possible branches. We thickened the grafted branch to distinguish it from the others.

Lemma 6.19. *Let $1 \leq k \leq n$ and $2 \leq n$, then*

$$\sum_{\tau \in \text{LPT}_{\mathbb{D},n,k}^0} \frac{2c(\tau, v_0^+(\tau)) - u(\tau, v_0^+(\tau)) - 1}{2(c(\tau, v_0^+(\tau)) - 1)} \prod_{u \in V(\tau,0)} c(\tau, u)^{-1} = 2^{n-1} n! k! S(n, k).$$

Proof. It suffices to show that

$$(6.7) \quad \sum_{\tau \in \text{LPT}_{\mathbb{D},n,k}^0} \left(1 + \frac{c(\tau, v_0^+(\tau)) - u(\tau, v_0^+(\tau))}{c(\tau, v_0^+(\tau)) - 1} \right) \text{Prob}(\tau) = k! S(n, k).$$

Define $f : \mathbb{T} \rightarrow \mathbb{Q}$ by setting

$$f(\sigma, \tau) = \begin{cases} \text{Prob}(\sigma) & \text{if } \sigma = \tau, \\ \frac{\text{Prob}(\sigma)}{c(\sigma, v_0^{++}(\sigma)) - 1} & \text{if } \sigma \sim \tau, \sigma \neq \tau, \\ 0 & \text{if } \sigma \not\sim \tau. \end{cases}$$

We already observed that, for each $\sigma \in \text{LPT}_{\mathbb{B},n,k}^0 \setminus \text{LPT}_{\mathbb{D},n,k}^0$, there are $c(\sigma, v_0^{++}(\sigma)) - 1$ trees $\tau \in \text{LPT}_{\mathbb{D},n,k}^0$ with $(\sigma, \tau) \in \mathcal{G}_{n,k}$. Thus Lemma 6.12 and (6.6) imply that

$$(6.8) \quad \begin{aligned} \sum_{(\sigma, \tau) \in \mathbb{T}} f(\sigma, \tau) &= \sum_{\substack{(\sigma, \tau) \in \mathbb{T} \\ \sigma \in \text{LPT}_{\mathbb{D},n,k}^0}} f(\sigma, \tau) + \sum_{\substack{(\sigma, \tau) \in \mathbb{T} \\ \sigma \notin \text{LPT}_{\mathbb{D},n,k}^0}} f(\sigma, \tau) \\ &= \sum_{\sigma \in \text{LPT}_{\mathbb{B},n,k}^0} \text{Prob}(\sigma) = k! S(n, k). \end{aligned}$$

For $\tau \in \text{LPT}_{\mathbb{D},n,k}^0$, set

$$\mathbf{g}(\tau) := c(\tau, v_0^+(\tau)) - u(\tau, v_0^+(\tau)) \in \mathbb{N}_0.$$

There are $2\mathbf{g}(\tau)$ different trees $\sigma \in \text{LPT}_{\mathbb{B},n,k}^0 \setminus \text{LPT}_{\mathbb{D},n,k}^0$ such that $(\sigma, \tau) \in \mathcal{G}_{n,k}$. To see this, we reverse the grafting operation: there are $\mathbf{g}(\tau)$ branches that can be cut and grafted onto the branch containing v_0 , on either the left or right side.

Therefore, if $g(\tau) = 0$, then $\sigma \sim \tau$ implies $\sigma = \tau$. Hence,

$$(6.9) \quad \sum_{\substack{(\sigma, \tau) \in T \\ g(\tau) = 0}} f(\sigma, \tau) = \sum_{\substack{\tau \in \text{LPT}_{D, n, k}^0 \\ g(\tau) = 0}} \text{Prob}(\tau).$$

If $g(\tau) > 0$, then $\sigma \sim \tau$ and $\sigma \neq \tau$ imply both $v_0^{++}(\sigma) = v_0^+(\tau)$ and $2\text{Prob}(\sigma) = \text{Prob}(\tau)$. Therefore,

$$(6.10) \quad \begin{aligned} \sum_{\substack{(\sigma, \tau) \in T \\ g(\tau) > 0}} f(\sigma, \tau) &= \sum_{\substack{(\sigma, \tau) \in T \\ g(\tau) > 0 \\ \sigma = \tau}} f(\sigma, \tau) + \sum_{\substack{(\sigma, \tau) \in T \\ g(\tau) > 0 \\ \sigma \neq \tau}} f(\sigma, \tau) \\ &= \sum_{\substack{\tau \in \text{LPT}_{D, n, k}^0 \\ g(\tau) > 0}} \text{Prob}(\tau) + \sum_{\tau \in \text{LPT}_{D, n, k}^0} \frac{g(\tau)\text{Prob}(\tau)}{c(\tau, v_0^+(\tau)) - 1}. \end{aligned}$$

We deduce (6.7) (and thus the lemma) by combining (6.8), (6.9), and (6.10):

$$k!S(n, k) = \sum_{(\sigma, \tau) \in T} f(\sigma, \tau) = \sum_{\tau \in \text{LPT}_{D, n, k}^0} \left(1 + \frac{c(\tau, v_0^+(\tau)) - u(\tau, v_0^+(\tau))}{c(\tau, v_0^+(\tau)) - 1} \right) \text{Prob}(\tau). \quad \square$$

6.3.3. *Proof of Theorem 6.10.* By Lemma 6.12,

$$|\text{LPT}_{X, n, k}| = \begin{cases} (n+1)!k!S(n, k) & \text{if } X = A, \\ 2^n(n+1)!k!S(n, k) & \text{if } X = B. \end{cases}$$

For each $F \in \Delta_{k-1}(\overline{\mathcal{L}}(X_n))$, choose $\tau_F \in \Phi_{X, n, k}^{-1}(F)$. By Lemma 6.15,

$$\begin{aligned} \sum_{F \in \Delta_{k-1}(\overline{\mathcal{L}}(X_n))} \pi_F(1) &= \sum_{F \in \Delta_{k-1}(\overline{\mathcal{L}}(X_n))} \left| \Phi_{X, n, k}^{-1}(F) \right| \left| \Omega_{X, n, k}^{-1}(\tau_F) \right| C_X(\tau_F) \\ &= \sum_{\tau \in \text{LPT}_{X, n, k}} C_X(\Omega_{X, n, k}(\tau)). \end{aligned}$$

If $X = A$, then

$$\sum_{\tau \in \text{LPT}_{A, n, k}} C_A(\Omega_{A, n, k}(\tau)) = |\text{LPT}_{A, n, k}| = \pi_{A_n}(1)k!S(n, k).$$

If $X = B$, then, by Lemma 6.17,

$$\sum_{\tau \in \text{LPT}_{B, n, k}} C_B(\Omega_{B, n, k}(\tau)) = \frac{|\text{LPT}_{B, n, k}|}{n+1} = \pi_{B_n}(1)k!S(n, k).$$

Lastly, if $X = D$, then by Lemma 6.19,

$$\sum_{\tau \in \text{LPT}_{D, n, k}} C_D(\Omega_{D, n, k}(\tau)) = \pi_{D_n}(1)k!S(n, k).$$

This completes the proof of Theorem 6.10. \square

6.4. **Proof of Theorem D.** Theorem 6.10 establishes Theorem D in the case of classical Coxeter arrangements. Formulae for the exceptional irreducible Coxeter arrangements not equal to E_8 are given in Appendix A.1.4. The proof of Theorem D for these arrangements follows by inspection of these formulae.

The case of general Coxeter arrangements with no irreducible factors isomorphic to E_8 follows now from Proposition 6.3. Indeed using, for instance, the *Carlitz identity* (cf. [21, Cor. 1.1])

$$\frac{E_n(T)}{(1-T)^{n+1}} = \sum_{k \geq 0} (k+1)^n T^k$$

for $n \geq 0$, one sees that for $n, m \geq 0$,

$$\frac{E_n(T)}{(1-T)^{n+1}} \star_T \frac{E_m(T)}{(1-T)^{m+1}} = \frac{E_{n+m}(T)}{(1-T)^{n+m+1}}.$$

This completes the proof of Theorem D. \square

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A. FURTHER EXAMPLES OF COARSE FLAG HILBERT–POINCARÉ SERIES

We collect explicit formulae for coarse flag Hilbert–Poincaré series of some hyperplane arrangements, including the irreducible Coxeter arrangements of rank at most seven in Section A.1. Most were computed using our package `HypIgu` [17] for SageMath [30]. Recall the notation (1.13) for the numerator of the coarse flag Hilbert–Poincaré series of a hyperplane arrangement \mathcal{A} . Throughout, let $X \in \{A, B, D\}$.

A.1. Irreducible Coxeter arrangements of rank at most seven.

A.1.1. Type A.

$$\begin{aligned}
 \mathcal{N}_{A_1}(Y, T) &= 1 + Y, \\
 \mathcal{N}_{A_2}(Y, T) &= 1 + 3Y + 2Y^2 + (2 + 3Y + Y^2)T, \\
 \mathcal{N}_{A_3}(Y, T) &= 1 + 6Y + 11Y^2 + 6Y^3 + (11 + 37Y + 37Y^2 + 11Y^3)T \\
 &\quad + (6 + 11Y + 6Y^2 + Y^3)T^2, \\
 \mathcal{N}_{A_4}(Y, T) &= 1 + 10Y + 35Y^2 + 50Y^3 + 24Y^4 \\
 &\quad + (47 + 260Y + 505Y^2 + 400Y^3 + 108Y^4)T \\
 &\quad + (108 + 400Y + 505Y^2 + 260Y^3 + 47Y^4)T^2 \\
 &\quad + (24 + 50Y + 35Y^2 + 10Y^3 + Y^4)T^3, \\
 \mathcal{N}_{A_5}(Y, T) &= 1 + 15Y + 85Y^2 + 225Y^3 + 274Y^4 + 120Y^5 \\
 &\quad + (197 + 1546Y + 4670Y^2 + 6700Y^3 + 4493Y^4 + 1114Y^5)T \\
 &\quad + (1268 + 7172Y + 15320Y^2 + 15320Y^3 + 7172Y^4 + 1268Y^5)T^2 \\
 &\quad + (1114 + 4493Y + 6700Y^2 + 4670Y^3 + 1546Y^4 + 197Y^5)T^3 \\
 &\quad + (120 + 274Y + 225Y^2 + 85Y^3 + 15Y^4 + Y^5)T^4, \\
 \mathcal{N}_{A_6}(Y, T) &= 1 + 21Y + 175Y^2 + 735Y^3 + 1624Y^4 + 1764Y^5 + 720Y^6 \\
 &\quad + (870 + 8918Y + 37163Y^2 + 80045Y^3 + 93065Y^4 + 54677Y^5 \\
 &\quad + 12542Y^6)T + (13184 + 100786Y + 309687Y^2 + 486220Y^3
 \end{aligned}$$

$$\begin{aligned}
& + 408786Y^4 + 174034Y^5 + 29383Y^6)T^2 + (29383 + 174034Y \\
& + 408786Y^2 + 486220Y^3 + 309687Y^4 + 100786Y^5 + 13184Y^6)T^3 \\
& + (12542 + 54677Y + 93065Y^2 + 80045Y^3 + 37163Y^4 + 8918Y^5 \\
& + 870Y^6)T^4 + (720 + 1764Y + 1624Y^2 + 735Y^3 + 175Y^4 \\
& + 21Y^5 + Y^6)T^5, \\
\mathcal{N}_{A_7}(Y, T) &= 1 + 28Y + 322Y^2 + 1960Y^3 + 6769Y^4 + 13132Y^5 + 13068Y^6 \\
& + 5040Y^7 + (4132 + 52669Y + 282471Y^2 + 823704Y^3 \\
& + 1403598Y^4 + 1387281Y^5 + 728999Y^6 + 155546Y^7)T \\
& + (134802 + 1301092Y + 5254088Y^2 + 11457173Y^3 \\
& + 14497658Y^4 + 10592498Y^5 + 4124012Y^6 + 659797Y^7)T^2 \\
& + (628282 + 4893332Y + 15809808Y^2 + 27375138Y^3 \\
& + 27375138Y^4 + 15809808Y^5 + 4893332Y^6 + 628282Y^7)T^3 \\
& + (659797 + 4124012Y + 10592498Y^2 + 14497658Y^3 \\
& + 11457173Y^4 + 5254088Y^5 + 1301092Y^6 + 134802Y^7)T^4 \\
& + (155546 + 728999Y + 1387281Y^2 + 1403598Y^3 + 823704Y^4 \\
& + 282471Y^5 + 52669Y^6 + 4132Y^7)T^5 + (5040 + 13068Y \\
& + 13132Y^2 + 6769Y^3 + 1960Y^4 + 322Y^5 + 28Y^6 + Y^7)T^6.
\end{aligned}$$

A.1.2. *Type B.*

$$\begin{aligned}
\mathcal{N}_{B_2}(Y, T) &= 1 + 4Y + 3Y^2 + (3 + 4Y + Y^2)T, \\
\mathcal{N}_{B_3}(Y, T) &= 1 + 9Y + 23Y^2 + 15Y^3 + (20 + 76Y + 76Y^2 + 20Y^3)T \\
& + (15 + 23Y + 9Y^2 + Y^3)T^2, \\
\mathcal{N}_{B_4}(Y, T) &= 1 + 16Y + 86Y^2 + 176Y^3 + 105Y^4 \\
& + (111 + 736Y + 1642Y^2 + 1376Y^3 + 359Y^4)T \\
& + (359 + 1376Y + 1642Y^2 + 736Y^3 + 111Y^4)T^2 \\
& + (105 + 176Y + 86Y^2 + 16Y^3 + Y^4)T^3 \\
\mathcal{N}_{B_5}(Y, T) &= 1 + 25Y + 230Y^2 + 950Y^3 + 1689Y^4 + 945Y^5 \\
& + (642 + 6146Y + 22220Y^2 + 36940Y^3 + 27058Y^4 + 6834Y^5)T \\
& + (5978 + 37082Y + 83660Y^2 + 83660Y^3 + 37082Y^4 + 5978Y^5)T^2 \\
& + (6834 + 27058Y + 36940Y^2 + 22220Y^3 + 6146Y^4 + 642Y^5)T^3 \\
& + (945 + 1689Y + 950Y^2 + 230Y^3 + 25Y^4 + Y^5)T^4, \\
\mathcal{N}_{B_6}(Y, T) &= 1 + 36Y + 505Y^2 + 3480Y^3 + 12139Y^4 + 19524Y^5 + 10395Y^6 \\
& + (4081 + 51460Y + 260329Y^2 + 669400Y^3 + 905659Y^4 \\
& + 592420Y^5 + 143211Y^6)T + (92476 + 793400Y + 2682964Y^2 \\
& + 4511120Y^3 + 3914404Y^4 + 1653560Y^5 + 268236Y^6)T^2 \\
& + (268236 + 1653560Y + 3914404Y^2 + 4511120Y^3 + 2682964Y^4 \\
& + 793400Y^5 + 92476Y^6)T^3 + (143211 + 592420Y + 905659Y^2 \\
& + 669400Y^3 + 260329Y^4 + 51460Y^5 + 4081Y^6)T^4 + (10395 \\
& + 19524Y + 12139Y^2 + 3480Y^3 + 505Y^4 + 36Y^5 + Y^6)T^5,
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_{\mathbb{B}_7}(Y, T) = & 1 + 49Y + 973Y^2 + 10045Y^3 + 57379Y^4 + 177331Y^5 \\
& + 264207Y^6 + 135135Y^7 + (28632 + 452376Y + 2970744Y^2 \\
& + 10468920Y^3 + 21232008Y^4 + 24456264Y^5 + 14475816Y^6 \\
& + 3329640Y^7)T + (1456493 + 15959421Y + 72251753Y^2 \\
& + 173850425Y^3 + 237761447Y^4 + 182794199Y^5 + 72699267Y^6 \\
& + 11564915Y^7)T^2 + (8886784 + 73960064Y + 250782336Y^2 \\
& + 445675776Y^3 + 445675776Y^4 + 250782336Y^5 + 73960064Y^6 \\
& + 8886784Y^7)T^3 + (11564915 + 72699267Y + 182794199Y^2 \\
& + 237761447Y^3 + 173850425Y^4 + 72251753Y^5 + 15959421Y^6 \\
& + 1456493Y^7)T^4 + (3329640 + 14475816Y + 24456264Y^2 \\
& + 21232008Y^3 + 10468920Y^4 + 2970744Y^5 + 452376Y^6 \\
& + 28632Y^7)T^5 + (135135 + 264207Y + 177331Y^2 + 57379Y^3 \\
& + 10045Y^4 + 973Y^5 + 49Y^6 + Y^7)T^6.
\end{aligned}$$

A.1.3. *Type D.*

$$\begin{aligned}
\mathcal{N}_{\mathbb{D}_4}(Y, T) = & 1 + 12Y + 50Y^2 + 84Y^3 + 45Y^4 \\
& + (67 + 396Y + 814Y^2 + 660Y^3 + 175Y^4)T \\
& + (175 + 660Y + 814Y^2 + 396Y^3 + 67Y^4)T^2 \\
& + (45 + 84Y + 50Y^2 + 12Y^3 + Y^4)T^3, \\
\mathcal{N}_{\mathbb{D}_5}(Y, T) = & 1 + 20Y + 150Y^2 + 520Y^3 + 809Y^4 + 420Y^5 \\
& + (397 + 3471Y + 11630Y^2 + 18250Y^3 + 12933Y^4 + 3239Y^5)T \\
& + (3143 + 18767Y + 41450Y^2 + 41450Y^3 + 18767Y^4 + 3143Y^5)T^2 \\
& + (3239 + 12933Y + 18250Y^2 + 11630Y^3 + 3471Y^4 + 397Y^5)T^3 \\
& + (420 + 809Y + 520Y^2 + 150Y^3 + 20Y^4 + Y^5)T^4, \\
\mathcal{N}_{\mathbb{D}_6}(Y, T) = & 1 + 30Y + 355Y^2 + 2100Y^3 + 6439Y^4 + 9390Y^5 + 4725Y^6 \\
& + (2539 + 29746Y + 141097Y^2 + 343660Y^3 + 445501Y^4 \\
& + 283234Y^5 + 67503Y^6)T + (50272 + 415232Y + 1363120Y^2 \\
& + 2246240Y^3 + 1931488Y^4 + 817568Y^5 + 134160Y^6)T^2 \\
& + (134160 + 817568Y + 1931488Y^2 + 2246240Y^3 + 1363120Y^4 \\
& + 415232Y^5 + 50272Y^6)T^3 + (67503 + 283234Y + 445501Y^2 \\
& + 343660Y^3 + 141097Y^4 + 29746Y^5 + 2539Y^6)T^4 + (4725 \\
& + 9390Y + 6439Y^2 + 2100Y^3 + 355Y^4 + 30Y^5 + Y^6)T^5, \\
\mathcal{N}_{\mathbb{D}_7}(Y, T) = & 1 + 42Y + 721Y^2 + 6510Y^3 + 33019Y^4 + 92358Y^5 + 127539Y^6 \\
& + 62370Y^7 + (17859 + 264979Y + 1645791Y^2 + 5526031Y^3 \\
& + 10759161Y^4 + 11991721Y^5 + 6930789Y^6 + 1570869Y^7)T \\
& + (804618 + 8519625Y + 37458806Y^2 + 88014129Y^3 \\
& + 118241942Y^4 + 89884515Y^5 + 35579114Y^6 + 5666211Y^7)T^2 \\
& + (4578872 + 37483512Y + 125642888Y^2 + 221947208Y^3 \\
& + 221947208Y^4 + 125642888Y^5 + 37483512Y^6 + 4578872Y^7)T^3
\end{aligned}$$

$$\begin{aligned}
& + (5666211 + 35579114Y + 89884515Y^2 + 118241942Y^3 \\
& + 88014129Y^4 + 37458806Y^5 + 8519625Y^6 + 804618Y^7)T^4 \\
& + (1570869 + 6930789Y + 11991721Y^2 + 10759161Y^3 \\
& + 5526031Y^4 + 1645791Y^5 + 264979Y^6 + 17859Y^7)T^5 \\
& + (62370 + 127539Y + 92358Y^2 + 33019Y^3 + 6510Y^4 \\
& + 721Y^5 + 42Y^6 + Y^7)T^6.
\end{aligned}$$

A.1.4. *Exceptional types.*

$$\begin{aligned}
\mathcal{N}_{E_6}(Y, T) &= 1 + 36Y + 510Y^2 + 3600Y^3 + 13089Y^4 + 22284Y^5 + 12320Y^6 \\
& + (4591 + 57420Y + 289824Y^2 + 748080Y^3 + 1020819Y^4 \\
& + 671940Y^5 + 162206Y^6)T + (103681 + 888840Y + 3011919Y^2 \\
& + 5080320Y^3 + 4411839Y^4 + 1858680Y^5 + 300401Y^6)T^2 \\
& + (300401 + 1858680Y + 4411839Y^2 + 5080320Y^3 + 3011919Y^4 \\
& + 888840Y^5 + 103681Y^6)T^3 + (162206 + 671940Y + 1020819Y^2 \\
& + 748080Y^3 + 289824Y^4 + 57420Y^5 + 4591Y^6)T^4 + (12320 \\
& + 22284Y + 13089Y^2 + 3600Y^3 + 510Y^4 + 36Y^5 + Y^6)T^5, \\
\mathcal{N}_{E_7}(Y, T) &= 1 + 63Y + 1617Y^2 + 21735Y^3 + 162939Y^4 + 663957Y^5 \\
& + 1286963Y^6 + 765765Y^7 + (90400 + 1553980Y + 11064984Y^2 \\
& + 42142884Y^3 + 92109360Y^4 + 113759940Y^5 + 70917656Y^6 \\
& + 16725596Y^7)T + (5577043 + 64210477Y + 304475955Y^2 \\
& + 763724661Y^3 + 1080226497Y^4 + 847444143Y^5 + 338480825Y^6 \\
& + 53381039Y^7)T^2 + (37767356 + 323700436Y + 1123040604Y^2 \\
& + 2022363924Y^3 + 2022363924Y^4 + 1123040604Y^5 \\
& + 323700436Y^6 + 37767356Y^7)T^3 + (53381039 + 338480825Y \\
& + 847444143Y^2 + 1080226497Y^3 + 763724661Y^4 + 304475955Y^5 \\
& + 64210477Y^6 + 5577043Y^7)T^4 + (16725596 + 70917656Y \\
& + 113759940Y^2 + 92109360Y^3 + 42142884Y^4 + 11064984Y^5 \\
& + 1553980Y^6 + 90400Y^7)T^5 + (765765 + 1286963Y + 663957Y^2 \\
& + 162939Y^3 + 21735Y^4 + 1617Y^5 + 63Y^6 + Y^7)T^6, \\
\mathcal{N}_{F_4}(Y, T) &= 1 + 24Y + 190Y^2 + 552Y^3 + 385Y^4 \\
& + (263 + 1992Y + 4994Y^2 + 4344Y^3 + 1079Y^4)T \\
& + (1079 + 4344Y + 4994Y^2 + 1992Y^3 + 263Y^4)T^2 \\
& + (385 + 552Y + 190Y^2 + 24Y^3 + Y^4)T^3, \\
\mathcal{N}_{G_2}(Y, T) &= 1 + 6Y + 5Y^2 + (5 + 6Y + Y^2)T, \\
\mathcal{N}_{H_2}(Y, T) &= 1 + 5Y + 4Y^2 + (4 + 5Y + Y^2)T, \\
\mathcal{N}_{H_3}(Y, T) &= 1 + 15Y + 59Y^2 + 45Y^3 \\
& + (44 + 196Y + 196Y^2 + 44Y^3)T \\
& + (45 + 59Y + 15Y^2 + Y^3)T^2, \\
\mathcal{N}_{H_4}(Y, T) &= 1 + 60Y + 1138Y^2 + 7140Y^3 + 6061Y^4
\end{aligned}$$

$$\begin{aligned}
& + (2099 + 20700Y + 63662Y^2 + 58500Y^3 + 13439Y^4)T \\
& + (13439 + 58500Y + 63662Y^2 + 20700Y^3 + 2099Y^4)T^2 \\
& + (6061 + 7140Y + 1138Y^2 + 60Y^3 + Y^4)T^3, \\
\mathcal{N}_{l_2(m)}(Y, T) & = 1 + mY + (m-1)Y^2 + (m-1 + mY + Y^2)T.
\end{aligned}$$

(For type $l_2(m)$, see also Proposition 4.2.)

A.2. Restrictions of type-D arrangements. Recall Definition 5.11 of the restrictions of type-D arrangements: for $n \geq 1$ and $m \in [n]_0$,

$$\mathcal{A}_{n,m} := D_n \cup \{X_k \mid k \in [n-m]\}.$$

We record the non-Coxeter restrictions up to rank four.

$$\begin{aligned}
\mathcal{N}_{\mathcal{A}_{3,2}}(Y, T) & = 1 + 7Y + 15Y^2 + 9Y^3 + (14 + 50Y + 50Y^2 + 14Y^3)T \\
& \quad + (9 + 15Y + 7Y^2 + Y^3)T^2, \\
\mathcal{N}_{\mathcal{A}_{3,1}}(Y, T) & = 1 + 8Y + 19Y^2 + 12Y^3 + (17 + 63Y + 63Y^2 + 17Y^3)T \\
& \quad + (12 + 19Y + 8Y^2 + Y^3)T^2, \\
\mathcal{N}_{\mathcal{A}_{4,3}}(Y, T) & = 1 + 13Y + 59Y^2 + 107Y^3 + 60Y^4 \\
& \quad + (78 + 481Y + 1021Y^2 + 839Y^3 + 221Y^4)T \\
& \quad + (221 + 839Y + 1021Y^2 + 481Y^3 + 78Y^4)T^2 \\
& \quad + (60 + 107Y + 59Y^2 + 13Y^3 + Y^4)T^3, \\
\mathcal{N}_{\mathcal{A}_{4,2}}(Y, T) & = 1 + 14Y + 68Y^2 + 130Y^3 + 75Y^4 \\
& \quad + (89 + 566Y + 1228Y^2 + 1018Y^3 + 267Y^4)T \\
& \quad + (267 + 1018Y + 1228Y^2 + 566Y^3 + 89Y^4)T^2 \\
& \quad + (75 + 130Y + 68Y^2 + 14Y^3 + Y^4)T^3, \\
\mathcal{N}_{\mathcal{A}_{4,1}}(Y, T) & = 1 + 15Y + 77Y^2 + 153Y^3 + 90Y^4 \\
& \quad + (100 + 651Y + 1435Y^2 + 1197Y^3 + 313Y^4)T \\
& \quad + (313 + 1197Y + 1435Y^2 + 651Y^3 + 100Y^4)T^2 \\
& \quad + (90 + 153Y + 77Y^2 + 15Y^3 + Y^4)T^3.
\end{aligned}$$

A.3. Shi arrangements. The *Shi arrangement* of type X_n is the (noncentral) hyperplane arrangement

$$SX_n = X_n \cup \{L-1 \mid L \in X_n\}.$$

A.3.1. *Type A.*

$$\begin{aligned}
\mathcal{N}_{SA_1}(Y, T) & = 1 + 2Y + T, \\
\mathcal{N}_{SA_2}(Y, T) & = 1 + 6Y + 9Y^2 + (10 + 24Y + 6Y^2)T + 4T^2 \\
\mathcal{N}_{SA_3}(Y, T) & = 1 + 12Y + 48Y^2 + 64Y^3 + (69 + 352Y + 524Y^2 + 160Y^3)T \\
& \quad + (151 + 380Y + 172Y^2 + 24Y^3)T^2 + 27T^3 \\
\mathcal{N}_{SA_4}(Y, T) & = 1 + 20Y + 150Y^2 + 500Y^3 + 625Y^4 \\
& \quad + (496 + 3905Y + 11045Y^2 + 12615Y^3 + 3955Y^4)T \\
& \quad + (3526 + 17220Y + 27140Y^2 + 14420Y^3 + 2510Y^4)T^2 \\
& \quad + (2931 + 7695Y + 4925Y^2 + 1305Y^3 + 120Y^4)T^3 + 256T^4.
\end{aligned}$$

(For SA_2 , see also Proposition 4.3.)

A.3.2. *Type B.*

$$\begin{aligned}
\mathcal{N}_{SB_2}(Y, T) &= 1 + 8Y + 16Y^2 + (16 + 44Y + 10Y^2)T + 9T^2 \\
\mathcal{N}_{SB_3}(Y, T) &= 1 + 18Y + 108Y^2 + 216Y^3 + (165 + 982Y + 1694Y^2 + 502Y^3)T \\
&\quad + (499 + 1370Y + 568Y^2 + 72Y^3)T^2 + 125T^3 \\
\mathcal{N}_{SB_4}(Y, T) &= 1 + 32Y + 384Y^2 + 2048Y^3 + 4096Y^4 \\
&\quad + (1912 + 17532Y + 57528Y^2 + 75596Y^3 + 24084Y^4)T \\
&\quad + (18806 + 100520Y + 169440Y^2 + 87848Y^3 + 14528Y^4)T^2 \\
&\quad + (20260 + 55436Y + 32928Y^2 + 8028Y^3 + 672Y^4)T^3 + 2401T^4.
\end{aligned}$$

A.3.3. *Type D.*

$$\begin{aligned}
\mathcal{N}_{SD_4}(Y, T) &= 1 + 24Y + 216Y^2 + 864Y^3 + 1296Y^4 \\
&\quad + (788 + 6612Y + 19960Y^2 + 24236Y^3 + 7600Y^4)T \\
&\quad + (6350 + 32320Y + 52456Y^2 + 27352Y^3 + 4616Y^4)T^2 \\
&\quad + (5964 + 15956Y + 9736Y^2 + 2460Y^3 + 216Y^4)T^3 + 625T^4.
\end{aligned}$$

A.4. **Catalan arrangements.** The *Catalan arrangement* of type X_n is the (non-central) hyperplane arrangement

$$CX_n = \{L + \varepsilon \mid L \in X_n, \varepsilon \in \{-1, 0, 1\}\}.$$

A.4.1. *Type A.*

$$\begin{aligned}
\mathcal{N}_{CA_1}(Y, T) &= 1 + 3Y + 2T, \\
\mathcal{N}_{CA_2}(Y, T) &= 1 + 9Y + 20Y^2 + (20 + 57Y + 13Y^2)T + 12T^2, \\
\mathcal{N}_{CA_3}(Y, T) &= 1 + 18Y + 107Y^2 + 210Y^3 \\
&\quad + (169 + 999Y + 1703Y^2 + 513Y^3)T \\
&\quad + (508 + 1377Y + 584Y^2 + 75Y^3)T^2 + 120T^3, \\
\mathcal{N}_{CA_4}(Y, T) &= 1 + 30Y + 335Y^2 + 1650Y^3 + 3024Y^4 \\
&\quad + (1597 + 14300Y + 45705Y^2 + 58420Y^3 + 18698Y^4)T \\
&\quad + (15138 + 79380Y + 131865Y^2 + 69180Y^3 + 11637Y^4)T^2 \\
&\quad + (15484 + 41890Y + 25495Y^2 + 6350Y^3 + 541Y^4)T^3 + 1680T^4.
\end{aligned}$$

A.4.2. *Type B.*

$$\begin{aligned}
\mathcal{N}_{CB_2}(Y, T) &= 1 + 12Y + 35Y^2 + (31 + 100Y + 21Y^2)T + 24T^2, \\
\mathcal{N}_{CB_3}(Y, T) &= 1 + 27Y + 239Y^2 + 693Y^3 \\
&\quad + (408 + 2756Y + 5352Y^2 + 1564Y^3)T \\
&\quad + (1583 + 4633Y + 1825Y^2 + 215Y^3)T^2 + 480T^3, \\
\mathcal{N}_{CB_4}(Y, T) &= 1 + 48Y + 854Y^2 + 6672Y^3 + 19305Y^4 \\
&\quad + (6399 + 65600Y + 237770Y^2 + 342784Y^3 + 110455Y^4)T \\
&\quad + (78975 + 447344Y + 787946Y^2 + 403792Y^3 + 64855Y^4)T^2 \\
&\quad + (98657 + 276896Y + 158262Y^2 + 36640Y^3 + 2857Y^4)T^3 \\
&\quad + 13440T^4.
\end{aligned}$$

A.4.3. *Type D.*

$$\begin{aligned}
\mathcal{N}_{\text{CD}_4}(Y, T) &= 1 + 36Y + 482Y^2 + 2844Y^3 + 6237Y^4 \\
&\quad + (2611 + 24636Y + 83158Y^2 + 112260Y^3 + 35767Y^4)T \\
&\quad + (27247 + 148140Y + 253006Y^2 + 130452Y^3 + 21379Y^4)T^2 \\
&\quad + (30669 + 84660Y + 49562Y^2 + 11916Y^3 + 985Y^4)T^3 + 3840T^4.
\end{aligned}$$

A.5. Generic central arrangements. Recall the generic central arrangements $\mathcal{U}_{n,m}$ for $n \leq m$ introduced in Section 4.6. We exemplify the general formula given in Proposition 6.8 with those for $m = n + 1$ and $n \in \{3, 4, 5, 6\}$. Note that $\mathcal{U}_{n,n+1}$ may be obtained by adding to the Boolean arrangement A_1^n (cf. Section 4.5) one additional hyperplane in general position; the one defined by the sum of coordinates will do. We also record formulae for $(n, m) \in \{(4, 7), (4, 8)\}$; see Section 6.2.2.

$$\begin{aligned}
\mathcal{N}_{\mathcal{U}_{3,4}}(Y, T) &= 1 + 4Y + 6Y^2 + 3Y^3 + (8 + 26Y + 26Y^2 + 8Y^3)T \\
&\quad + (3 + 6Y + 4Y^2 + Y^3)T^2, \\
\mathcal{N}_{\mathcal{U}_{4,5}}(Y, T) &= 1 + 5Y + 10Y^2 + 10Y^3 + 4Y^4 \\
&\quad + (22 + 100Y + 170Y^2 + 125Y^3 + 33Y^4)T \\
&\quad + (33 + 125Y + 170Y^2 + 100Y^3 + 22Y^4)T^2 \\
&\quad + (4 + 10Y + 10Y^2 + 5Y^3 + Y^4)T^3, \\
\mathcal{N}_{\mathcal{U}_{4,7}}(Y, T) &= 1 + 7Y + 21Y^2 + 35Y^3 + 20Y^4 \\
&\quad + (60 + 315Y + 609Y^2 + 483Y^3 + 129Y^4)T \\
&\quad + (129 + 483Y + 609Y^2 + 315Y^3 + 60Y^4)T^2 \\
&\quad + (20 + 35Y + 21Y^2 + 7Y^3 + Y^4)T^3, \\
\mathcal{N}_{\mathcal{U}_{4,8}}(Y, T) &= 1 + 8Y + 28Y^2 + 56Y^3 + 35Y^4 \\
&\quad + (89 + 488Y + 980Y^2 + 792Y^3 + 211Y^4)T \\
&\quad + (211 + 792Y + 980Y^2 + 488Y^3 + 89Y^4)T^2 \\
&\quad + (35 + 56Y + 28Y^2 + 8Y^3 + Y^4)T^3, \\
\mathcal{N}_{\mathcal{U}_{5,6}}(Y, T) &= 1 + 6Y + 15Y^2 + 20Y^3 + 15Y^4 + 5Y^5 \\
&\quad + (52 + 297Y + 685Y^2 + 795Y^3 + 459Y^4 + 104Y^5)T \\
&\quad + (198 + 1023Y + 2085Y^2 + 2085Y^3 + 1023Y^4 + 198Y^5)T^2 \\
&\quad + (104 + 459Y + 795Y^2 + 685Y^3 + 297Y^4 + 52Y^5)T^3 \\
&\quad + (5 + 15Y + 20Y^2 + 15Y^3 + 6Y^4 + Y^5)T^4, \\
\mathcal{N}_{\mathcal{U}_{6,7}}(Y, T) &= 1 + 7Y + 21Y^2 + 35Y^3 + 35Y^4 + 21Y^5 + 6Y^6 + (114 + 777Y \\
&\quad + 2233Y^2 + 3465Y^3 + 3059Y^4 + 1449Y^5 + 285Y^6)T + (906 \\
&\quad + 5796Y + 15400Y^2 + 21700Y^3 + 17052Y^4 + 7070Y^5 + 1208Y^6)T^2 \\
&\quad + (1208 + 7070Y + 17052Y^2 + 21700Y^3 + 15400Y^4 + 5796Y^5 \\
&\quad + 906Y^6)T^3 + (285 + 1449Y + 3059Y^2 + 3465Y^3 + 2233Y^4 \\
&\quad + 777Y^5 + 114Y^6)T^4 + (6 + 21Y + 35Y^2 + 35Y^3 + 21Y^4 \\
&\quad + 7Y^5 + Y^6)T^5.
\end{aligned}$$

A.6. Resonance arrangements. For $n \in \mathbb{N}$, the *resonance arrangement* is

$$\mathcal{R}_n = \left\{ \sum_{i \in I} X_i \mid \emptyset \neq I \subseteq [n] \right\}.$$

For $n \leq 2$, $\mathcal{R}_n \cong \mathbf{A}_n$, given in Section A.1.1, and $\mathcal{R}_3 \cong \mathcal{A}_{3,2}$, given in Section A.2.

$$\mathcal{N}_{\mathcal{R}_4}(Y, T) = 1 + 15Y + 80Y^2 + 170Y^3 + 104Y^4$$

$$\begin{aligned} &+ (112 + 730Y + 1630Y^2 + 1365Y^3 + 353Y^4)T \\ &+ (353 + 1365Y + 1630Y^2 + 730Y^3 + 112Y^4)T^2 \\ &+ (104 + 170Y + 80Y^2 + 15Y^3 + Y^4)T^3, \end{aligned}$$

$$\mathcal{N}_{\mathcal{R}_5}(Y, T) = 1 + 31Y + 375Y^2 + 2130Y^3 + 5270Y^4 + 3485Y^5 + (1782 + 17817Y$$

$$\begin{aligned} &+ 68375Y^2 + 121745Y^3 + 92659Y^4 + 23254Y^5)T + (18818 \\ &+ 120923Y + 280775Y^2 + 280775Y^3 + 120923Y^4 + 18818Y^5)T^2 \\ &+ (23254 + 92659Y + 121745Y^2 + 68375Y^3 + 17817Y^4 + 1782Y^5)T^3 \\ &+ (3485 + 5270Y + 2130Y^2 + 375Y^3 + 31Y^4 + Y^5)T^4, \end{aligned}$$

$$\mathcal{N}_{\mathcal{R}_6}(Y, T) = 1 + 63Y + 1652Y^2 + 22435Y^3 + 159460Y^4 + 510524Y^5 + 371909Y^6$$

$$\begin{aligned} &+ (77254 + 1088220Y + 6136361Y^2 + 17666495Y^3 + 26863403Y^4 \\ &+ 19032139Y^5 + 4709836Y^6)T + (2362293 + 21610148Y \\ &+ 77625345Y^2 + 137120970Y^3 + 121428629Y^4 + 50644846Y^5 \\ &+ 7959697Y^6)T^2 + (7959697 + 50644846Y + 121428629Y^2 \\ &+ 137120970Y^3 + 77625345Y^4 + 21610148Y^5 + 2362293Y^6)T^3 \\ &+ (4709836 + 19032139Y + 26863403Y^2 + 17666495Y^3 + 6136361Y^4 \\ &+ 1088220Y^5 + 77254Y^6)T^4 + (371909 + 510524Y + 159460Y^2 \\ &+ 22435Y^3 + 1652Y^4 + 63Y^5 + Y^6)T^5. \end{aligned}$$

A.7. 2-sum arrangements. For $n \geq 3$, the *2-sum arrangement* is the central hyperplane arrangement

$$\mathcal{S}_n = \mathbf{A}_n \cup \{X_i + X_j - X_k - X_\ell \mid \text{distinct } i, j, k, \ell \in [n+1]\}.$$

$$\mathcal{N}_{\mathcal{S}_3}(Y, T) = 1 + 9Y + 23Y^2 + 15Y^3$$

$$\begin{aligned} &+ (20 + 76Y + 76Y^2 + 20Y^3)T \\ &+ (15 + 23Y + 9Y^2 + Y^3)T^2, \end{aligned}$$

$$\mathcal{N}_{\mathcal{S}_4}(Y, T) = 1 + 25Y + 215Y^2 + 695Y^3 + 504Y^4$$

$$\begin{aligned} &+ (342 + 2605Y + 6625Y^2 + 5795Y^3 + 1433Y^4)T \\ &+ (1433 + 5795Y + 6625Y^2 + 2605Y^3 + 342Y^4)T^2 \\ &+ (504 + 695Y + 215Y^2 + 25Y^3 + Y^4)T^3, \end{aligned}$$

$$\mathcal{N}_{\mathcal{S}_5}(Y, T) = 1 + 60Y + 1360Y^2 + 14010Y^3 + 59119Y^4 + 46410Y^5 + (12332$$

$$\begin{aligned} &+ 147046Y + 656060Y^2 + 1328440Y^3 + 1075448Y^4 + 268354Y^5)T \\ &+ (179963 + 1253627Y + 3070730Y^2 + 3070730Y^3 + 1253627Y^4 \\ &+ 179963Y^5)T^2 + (268354 + 1075448Y + 1328440Y^2 + 656060Y^3 \\ &+ 147046Y^4 + 12332Y^5)T^3 + (46410 + 59119Y + 14010Y^2 \\ &+ 1360Y^3 + 60Y^4 + Y^5)T^4. \end{aligned}$$

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