

TENSOR ISOMORPHISM BY CONJUGACY OF LIE ALGEBRAS

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ABSTRACT. We introduce an algorithm to decide isomorphism between tensors. The algorithm uses the Lie algebra of derivations of a tensor to compress the space in which the search takes place to a so-called *tensor space*. To make the method practicable we give a polynomial-time algorithm to solve a generalization of module isomorphism for a common class of Lie modules. As a consequence, we show that isomorphism testing is in polynomial time for tensors whose derivation algebras are classical Lie algebras and whose tensor spaces are 1-dimensional. The method has been implemented in the MAGMA computer algebra system.

1. INTRODUCTION

Techniques to decide isomorphism for algebraic structures such as groups, algebras, and modules often involve testing whether two multilinear maps (tensors) are equal under basis changes. Examples include isomorphism tests for finite p -groups that work through the factors of the exponent p -central series [13, 28], and others that use more general filtrations [27, 31]. A recent rethinking of these approaches led to an isomorphism test for graded algebras that identifies an optimal route through the filtration [8]. In all of these techniques, the initial task is to decide isomorphism between tensors. This paper introduces a new algorithm to solve tensor isomorphism by exploiting the action of the Lie algebra of derivations on the vector space of tensors. The algorithm is particularly well suited to isomorphism problems for highly symmetric structures, such as those found in [1, 15, 16]. We note that methods for generic tensors, algebras, and groups can be found in [3, 13, 25].

1.1. Tensor isomorphism. Fix a field K and finite-dimensional K -vector spaces V_1, \dots, V_ℓ . A *tensor* is an element, t , of $(V_1 \otimes \dots \otimes V_\ell)^* = \text{hom}_K(V_1 \otimes \dots \otimes V_\ell, K)$, and the integer $\ell \geq 2$ is its *valence*. We interpret t as a multilinear function $\langle t | : V_1 \times \dots \times V_\ell \rightarrow K$ (\rightarrow to denote multilinear) that may be evaluated on inputs $v = (v_1, \dots, v_\ell) \in V_1 \times \dots \times V_\ell$ using a Dirac styled “bra-ket” notation as follows:

$$\langle t | v \rangle = \langle t | v_1, \dots, v_\ell \rangle \in K.$$

A tensor is also determined by its coordinates relative to bases $\{e_{a_1}, \dots, e_{a_\ell}\}$ for each V_a . That is, for $a \in [\ell] = \{1, \dots, \ell\}$ and $i_a \in [d_a]$, one specifies the scalar

$$(1.1) \quad T_{i_1 \dots i_\ell} = \langle t | e_{1i_1}, \dots, e_{\ell i_\ell} \rangle \in K.$$

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As in elementary linear algebra, one can pass back and forth between the “hypermatrix” $[T_i]_{i \in [d_1] \times \dots \times [d_\ell]} \in K^{d_1 \times \dots \times d_\ell}$ and the associated multilinear map $\langle t |$.

For $\omega = (\omega_1, \dots, \omega_\ell) \in \text{GL}(V_1) \times \dots \times \text{GL}(V_\ell)$ and $t \in (V_1 \otimes \dots \otimes V_\ell)^*$, define $t^\omega \in (V_1 \otimes \dots \otimes V_\ell)^*$ as follows:

$$\langle t^\omega | v \rangle = \langle t | \omega v \rangle = \langle t | \omega_1 v_1, \dots, \omega_\ell v_\ell \rangle.$$

Tensors s and t are *isomorphic* if $s^\omega = t$ for some $\omega \in \text{GL}(V_1) \times \dots \times \text{GL}(V_\ell)$. Expressed in terms of hypermatrices, this is a natural extension of equivalence of matrices up to row and column operations. In a recent paper, Grochow and Qiao prove that the problem of deciding whether two tensors are isomorphic has connections with many familiar and difficult decision problems, such as the graph and group isomorphism problems [18].

Throughout the paper ‘algorithm’ will mean *Las Vegas algorithm*, which always returns the correct answer but may, with bounded probability, abort without an answer. We adopt an *arithmetic model* of computation, wherein all field operations in K have unit cost and are precise (no rounding). When K is infinite, we assume the existence of oracles to factor elements from \mathbb{Z} and from $\mathbb{Q}[x]$. Note that the product $\cdot : A \times A \rightarrow A$ of a K -algebra can be treated as a tensor in $(A \otimes A \otimes A)^*$. Hence, we assume tensors and algebras are both given by fixing bases for all vector spaces involved, and specifying the scalars in (1.1). (In the algebra context it is common to refer to the scalars as *structure constants*.)

1.2. The derivation-densor method. Write $\mathfrak{gl}(V)$ for the Lie algebra on $\text{End}(V)$ with Lie bracket given by the matrix commutator. The *derivation algebra* of a tensor $t \in (V_1 \otimes \dots \otimes V_\ell)^*$ is the Lie algebra

$$\text{Der}(t) = \left\{ \delta \in \bigoplus_{a=1}^{\ell} \mathfrak{gl}(V_a) \mid \forall a \in [\ell], \forall v_a \in V_a, \langle t | \delta_1 v_1, \dots, v_\ell \rangle + \dots + \langle t | v_1, \dots, \delta_\ell v_\ell \rangle = 0 \right\}.$$

Next we use a generalization of the standard Whitney tensor products called the *densor spaces* (short for *derivation tensor spaces*) as introduced in [14]. Start with $\Delta \subset \mathfrak{gl}(V_1) \times \dots \times \mathfrak{gl}(V_\ell)$ and define

$$\mathfrak{D}V_1, \dots, V_\ell \mathfrak{D}\Delta = \{s \in (V_1 \otimes \dots \otimes V_\ell)^* \mid \Delta \subseteq \text{Der}(s)\}.$$

The notation $\mathfrak{D}V_1, \dots, V_\ell \mathfrak{D}\Delta$ perhaps requires some explanation. Consider the case $\ell = 2$, and let $\delta \mapsto \delta^\dagger$ be a ring anti-isomorphism $\text{End}(V_1) \rightarrow \text{End}(V_1)^{op}$. If $\hat{\Delta} = \{(-\delta_1^\dagger, \delta_2) \mid (\delta_1, \delta_2) \in \Delta\} \subset \text{End}(V_1)^{op} \times \text{End}(V_2)$, then

$$\begin{aligned} (V_1 \otimes_{\hat{\Delta}} V_2)^* &= \{t \in (V_1 \otimes V_2)^* \mid \langle t | v_1(-\delta_1^\dagger), v_2 \rangle = \langle t | v_1, \delta_2 v_2 \rangle\} \\ &= \{t \in (V_1 \otimes V_2)^* \mid \langle t | \delta_1 v_1, v_2 \rangle + \langle t | v_1, \delta_2 v_2 \rangle = 0\} = \mathfrak{D}V_1, V_2 \mathfrak{D}\Delta. \end{aligned}$$

Thus, densor spaces are multivalent generalizations of \otimes . The notation of $\mathfrak{D} \mathfrak{D}$ is conceived as backward and forward letters D —for derivation—but stylized to evoke the connection to the symbol \otimes . We abbreviate $\mathfrak{D}V_1, \dots, V_\ell \mathfrak{D}\text{Der}(t)$ to $\mathfrak{D}t \mathfrak{D}$.

Our algorithm for tensor isomorphism uses $\text{Der}(t)$ and $\mathfrak{D}t \mathfrak{D}$ together with the subgroup of $\text{GL}(V_1) \times \dots \times \text{GL}(V_\ell)$ that normalizes $\text{Der}(t)$, namely

$$N(\text{Der}(t)) = \left\{ \omega \in \prod_{a=1}^{\ell} \text{GL}(V_a) \mid \forall \delta \in \text{Der}(t), (\omega_1^{-1} \delta_1 \omega_1, \dots, \omega_\ell^{-1} \delta_\ell \omega_\ell) \in \text{Der}(t) \right\}.$$

Algorithm 1 gives a high level description of the isomorphism test, which we call the *derivation-densor method*.

Algorithm 1 (Derivation–Densor)

Input: Tensors $s, t \in (V_1 \otimes \cdots \otimes V_\ell)^*$.

Output: $\omega \in G := \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_\ell)$ with $s^\omega = t$, if such exists.

- 1: Compute $\mathrm{Der}(s)$ and $\mathrm{Der}(t)$.
 - 2: **if** $(\exists \mu \in G)(\mathrm{Der}(s)^\mu = \mathrm{Der}(t))$ **then**
 - 3: Build the densor space $\mathfrak{D}(t)$.
 - 4: Compute the action of $N(\mathrm{Der}(t))$ on $\mathfrak{D}(t)$.
 - 5: **if** $(\exists \nu \in N(\mathrm{Der}(t)))(s^{\mu\nu} = t)$ **then return** $\omega := \mu\nu$.
 - 6: **else** Report $s \not\cong t$.
 - 7: **else** Report $s \not\cong t$.
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Lines 1 and 3 are carried out by solving systems of linear equations. The practicability of the method depends critically on the related problems in Lines 2, 4 and 5. These problems are known to be difficult in general, but if we have sufficient control of $\mathrm{Der}(t)$ and its representation in $\mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_\ell)$, then we can carry out these tasks directly in polynomial time (cf. Theorem 1.4).

We note that existing methods [5, 6, 10, 24, 32] employ similar strategies, but instead of Lie algebras of derivations they used associative algebras A_i of so-called *adjoints*, and instead of the densor space they compress the search space using more traditional tensors of the form $(V_1 \otimes_{A_1} \cdots \otimes_{A_{\ell-1}} V_\ell)^*$. The associative approach became known as the *adjoint-tensor method*; details are given in Section 2.

The use of Lie algebras has a distinctive advantage over adjoint-tensor methods. Simple modules of associative algebras have fixed dimensions; for example, K^2 is the only simple module of $\mathbb{M}_2(K)$. Thus, if the adjoint algebras A_i have bounded dimensions, then $\dim(V_1 \otimes_{A_1} \cdots \otimes_{A_{\ell-1}} V_\ell)^*$ is proportional to $\prod_{i=1}^{\ell} \dim V_i$. As $\mathfrak{D}(t)$ is a Lie module, its dimension is governed by quantities such as the Littlewood–Richardson coefficients. Even Lie algebras of bounded dimension, such as $\mathfrak{sl}_2(K)$, can have simple modules of arbitrary dimensions, which means that $\dim \mathfrak{D}(t)$ can be surprisingly small. Indeed, there are infinite families of tensors t with $\dim \mathfrak{D}(t) = 1$ such that, if A_i are associative algebras satisfying $t \in (V_1 \otimes_{A_1} \cdots \otimes_{A_{\ell-1}} V_\ell)^*$, then $\dim(V_1 \otimes_{A_1} \cdots \otimes_{A_{\ell-1}} V_\ell)^* = \prod_{i=1}^{\ell} \dim V_i$ (cf. Theorem 4.6 and Remark 5.1).

As noted above, the performance of derivation-densor depends, for certain inputs, on difficulties faced in lines 2 and 4. It is often line 5, however, that presents the most serious challenges. Here, we search through the cosets of the centralizer $C(\mathrm{Der}(t)) = \{\omega \in N(\mathrm{Der}(t)) \mid \delta \in \mathrm{Der}(t), \delta^\omega = \delta\}$ in $N(\mathrm{Der}(t))$. Although efficient module-theoretic techniques can often be used to refine this search, for many tensors t the derivation algebra $\mathrm{Der}(t) = \{(-\alpha_1, \dots, -\alpha_{\ell-1}, \alpha_1 + \cdots + \alpha_\ell) \mid \alpha_a \in K\}$ is as small as it can possibly be, which in turn means $N(\mathrm{Der}(t))/C(\mathrm{Der}(t))$ can be as large as $\mathrm{PGL}(V_1) \times \cdots \times \mathrm{PGL}(V_\ell)$. Broadly speaking, derivation-densor works best when the given tensors possess enough global symmetry to be detectable by their derivation algebras, and such settings are indeed the focus of this paper. However, since setting it up involves only linear algebra, derivation-densor serves as an efficient first reduction for many isomorphism problems.

1.3. Using the derivation-densor method. The key problems in Lines 2 and 4 of Algorithm 1 reduce to a variation of the module isomorphism problem for (non-associative) algebras. Write ${}_A V$ to indicate that V is a left A -module. We say

modules ${}_{A_1}V_1$ and ${}_{A_2}V_2$ are *pseudo-isomorphic* if there is an algebra isomorphism $\psi: A_1 \rightarrow A_2$ and a K -linear isomorphism $\Psi: V_1 \rightarrow V_2$ such that

$$(1.2) \quad (\forall x_1 \in A_1)(\forall v_1 \in V_1) \quad \Psi(x_1 v_1) = \psi(x_1)\Psi(v_1).$$

When $A = A_1 = A_2$ and $\psi = 1$ this is the usual notion of A -module isomorphism. In that case, a polynomial-time algorithm of the first author and Luks [4] may be used to build Ψ from units in the associative algebra $\text{Hom}_L(V_1, V_2)\text{Hom}_L(V_2, V_1) \subset \text{End}_L(V_1)$. When ψ is allowed to vary, however, the problem becomes much more difficult. Indeed, Grochow has shown that deciding Lie module pseudo-isomorphism is at least as hard as deciding graph isomorphism [17]. There are, however, polynomial-time solutions for special classes, such as modules of simple and cyclic associative algebras [11], and simple modules of simple Lie algebras over \mathbb{C} [17]. The following theorem, which we consider to be of independent interest, supplements those results (a Lie algebra L has *Chevalley type* if $[L, L]$ has a Chevalley basis).

Theorem 1.3. *Let K be a field with either $K = \mathbb{F}_q$ finite or K/\mathbb{Q} finite. There is a polynomial-time algorithm that decides pseudo-isomorphism of faithful simple finite-dimensional Lie modules over Lie algebras of Chevalley type.*

A tensor $t \in (V_1 \otimes \cdots \otimes V_\ell)^*$ is *degenerate* if there exists $0 \neq v_a \in V_a$ such that, for all $b \neq a$ and all $v_b \in V_b$, $\langle t|v_1, \dots, v_\ell \rangle = 0$. It is elementary to reduce the isomorphism problem for arbitrary tensors to the nondegenerate case. The polynomial-time algorithm in Theorem 1.3 is vital to the proof of the next theorem.

Theorem 1.4. *Let K be a field with either $K = \mathbb{F}_q$ finite or K/\mathbb{Q} finite, let V_1, \dots, V_ℓ be finite-dimensional K -spaces, and let $d = \sum_{a=1}^{\ell} \dim_K V_a$. For nondegenerate $s, t \in (V_1 \otimes \cdots \otimes V_\ell)^*$ with $\text{Der}(s)$ of Chevalley type and $\dim \langle s \rangle = 1$, one can decide using $d^{O(1)}$ steps whether $s \cong t$.*

The tensors t for which $\dim \langle t \rangle = 1$ are interesting special cases in their own right, and they are more common than one might suspect. In Section 4 we construct an infinite family of tensors, arising naturally from the representation theory of classical Lie algebras, whose densor space is 1-dimensional.

2. ALGEBRAIC TENSOR COMPRESSION

The use of rings to decrease the dimension of the search spaces in tensor isomorphism is not new, but it has heretofore been developed only for associative rings. In this section we briefly describe the history of *algebraic tensor compression*, and how it led to the derivation-densor method.

2.1. The adjoint-tensor method. The first tensor compression method was introduced in [24] for the case $\ell = 2$, and soon after generalized in [6, 10, 32]. Given a bilinear map (bimap) $\langle t|: V_1 \times V_2 \rightarrow V_0$, its *adjoint algebra* is

$$\text{Adj}(t) = \{\alpha \in \text{End}(V_1) \times \text{End}(V_2)^{\text{op}} \mid \langle t|\alpha_2 v_2, v_1 \rangle = \langle t|v_2, v_1 \alpha_1 \rangle\},$$

and its associated tensor space is $V_1 \otimes_{\text{Adj}(t)} V_2$. Observe that $\langle t|$ naturally factors through the bimap $\otimes_{\text{Adj}(t)}: V_1 \times V_2 \rightarrow V_1 \otimes_{\text{Adj}(t)} V_2$.

The *adjoint-tensor* method solves the isomorphism problem between bimaps s and t by first deciding if there exists μ conjugating $\text{Adj}(s)$ to $\text{Adj}(t)$, which is done by the methods of [9, 22]. Then we carry out a search within the compressed space

$\text{Hom}(V_1 \otimes_{\text{Adj}(t)} V_2, V_0)$ —in which both s^μ and t now reside—under the action of the potentially much smaller group normalizing $\text{Adj}(t)$, modulo $\text{Adj}(t)^\times$.

This process is captured concisely as follows:

$$(\exists \varphi)(s^\varphi = t) \iff (\exists \mu)(\exists \nu) \begin{cases} \text{Adj}(s)^\mu & := \mu^{-1} \text{Adj}(s)\mu = \text{Adj}(t), \\ \text{Adj}(t)^\nu & = \text{Adj}(t), \text{ and} \\ (s^\mu)^\nu & = t \in \text{Hom}(V_1 \otimes_{\text{Adj}(t)} V_2, V_0). \end{cases}$$

The method distinguishes V_0 due to its role as the codomain. One could, however, just as easily consider s, t as tensors in $(V_1 \otimes V_2 \otimes V_0^*)^*$. With this interpretation the compressed tensor space is $(V_1 \otimes_{\text{Adj}(t)} V_2 \otimes V_0^*)^*$, which now seems like an arbitrary choice. To reconcile the apparent asymmetry, the third author introduced a generalization involving operations between all *pairs* of V_a and V_b [32]. *The guiding principle of the derivation-densor algorithm is to move away from binary tensor products entirely.*

2.2. A broader view. Using an emerging theory of transverse operators on tensor spaces, one can generalize the adjoint-tensor method in many different ways. The theory, which is based on a ternary Galois correspondence, is developed in detail in the forthcoming work [14]; we describe and prove here only what is needed for our isomorphism test.

Given $t \in T := (V_1 \otimes \cdots \otimes V_\ell)^*$, $f(X) = \sum_e \lambda_e X^e$ an element of the polynomial ring $K[X] := K[x_1, \dots, x_\ell]$, and $\omega \in \Omega := \text{End}(V_1) \times \cdots \times \text{End}(V_\ell)$, define a new multilinear form $\langle t|f(\omega) \rangle$ as follows:

$$\langle t|f(\omega)|v \rangle = \sum_e \lambda_e \langle t|\omega_1^{e_1} v_1, \dots, \omega_\ell^{e_\ell} v_\ell \rangle.$$

Given $S \subset T$, $P \subset K[X]$, and $\Upsilon \subset \Omega$, define three sets

$$\begin{aligned} \mathcal{N}(P, \Upsilon) &= \{t \in T \mid \forall f \in P, \forall \omega \in \Upsilon, \forall v, \langle t|f(\omega)|v \rangle = 0\}, \\ \mathcal{I}(S, \Upsilon) &= \{f \in K[X] \mid \forall t \in S, \forall \omega \in \Upsilon, \forall v, \langle t|f(\omega)|v \rangle = 0\}, \\ \mathcal{Z}(S, P) &= \{\omega \in \Omega \mid \forall t \in S, \forall f \in P, \forall v, \langle t|f(\omega)|v \rangle = 0\}. \end{aligned}$$

Then $\mathcal{N}(P, \Upsilon)$ is a subspace, $\mathcal{I}(S, \Upsilon)$ is an ideal, and $\mathcal{Z}(S, P)$ is an algebraic set, and they satisfy the following Galois correspondence property:

$$(2.1) \quad S \subset \mathcal{N}(P, \Upsilon) \iff P \subset \mathcal{I}(S, \Upsilon) \iff \Upsilon \subset \mathcal{Z}(S, P).$$

For an algebraic perspective, each $\omega \in \Omega$ defines a representation $\rho_\omega: K[X] \rightarrow \text{End}(T)$, where $f \mapsto (\langle t| \mapsto \langle t|f(\omega) \rangle)$. The sets $\mathcal{I}(S, \Upsilon) = \bigcap_{\omega \in \Upsilon} \mathcal{I}(S, \omega)$ are annihilator ideals in $K[X]$, so they are a multilinear generalization of the concept of minimal polynomials. The sets $\mathcal{N}(P, \Upsilon)$ generalize tensor products. For example, if $A \subseteq \text{End}(V_1) \times \text{End}(V_2)^{\text{op}}$ is an associative algebra and $f(X) = x_1 - x_2$, then $\mathcal{N}(x_1 - x_2, A)$ is the usual tensor product $(V_1 \otimes_A V_2)^*$. For, if $t \in (V_1 \otimes_A V_2)^*$ and all $(\varphi_1, \varphi_2) \in A$, then

$$\langle t|f(\varphi_1, \varphi_2)|v_1, v_2 \rangle = \langle t|\varphi_1 v_1, v_2 \rangle - \langle t|v_1, \varphi_2 v_2 \rangle = 0.$$

The algebraic sets $\mathcal{Z}(S, P)$ may, depending on P , come equipped with algebraic structure external to their definition. For example,

$$\text{Adj}(S) = \mathcal{Z}(S, x_1 - x_2) \quad \text{Der}(S) = \mathcal{Z}(S, x_1 + \cdots + x_\ell)$$

are, respectively, associative and Lie algebras.

The Galois correspondence in (2.1) relating these three constructions has closures. For example, if $\mathbf{d} = x_1 + \cdots + x_\ell$, then

$$\langle t \rangle = \mathcal{N}(\mathbf{d}, \text{Der}(t)) = \mathcal{N}(\mathbf{d}, \mathcal{Z}(t, \mathbf{d})).$$

The adjoint-tensor uses a different closure:

$$(V_1 \otimes_{\text{Adj}(t)} V_2 \otimes V_0^*)^* = \mathcal{N}(x_1 - x_2, \mathcal{Z}(t, x_1 - x_2)).$$

In fact, the following proposition elucidates a tensor compression method for *every* ideal in $K[X]$! Note, $\Omega^\times = \text{GL}(V_1) \times \cdots \times \text{GL}(V_\ell)$ is the group of units of Ω .

Proposition 2.2. *Let $s, t \in (V_1 \otimes \cdots \otimes V_\ell)^*$ and let P be an ideal of $K[X]$. Then there exists $\varphi \in \Omega^\times$ such that $s^\varphi = t$ if, and only if, there exist $\mu, \nu \in \Omega$ such that*

$$\begin{aligned} \mathcal{Z}(s, P)^\mu &= \mathcal{Z}(t, P), \\ \mathcal{Z}(t, P)^\nu &= \mathcal{Z}(t, P), \text{ and} \\ (s^\mu)^\nu &= t \in \mathcal{N}(P, \mathcal{Z}(t, P)). \end{aligned}$$

Proof. Suppose $\varphi \in \Omega^\times$ satisfies $s^\varphi = t$. Fix $\omega \in \mathcal{Z}(s^\varphi, P)$ and $f = \sum_e \lambda_e X^e \in P$. Then, for all $v_a \in V_a$,

$$\begin{aligned} 0 &= \langle s^\varphi | f(\omega) | v_1, \dots, v_\ell \rangle = \sum_e \lambda_e \langle s^\varphi | \omega_1^{e_1} v_1, \dots, \omega_\ell^{e_\ell} v_\ell \rangle \\ &= \sum_e \lambda_e \langle s | \varphi_1 \omega_1^{e_1} \varphi_1^{-1} u_1, \dots, \varphi_\ell \omega_\ell^{e_\ell} \varphi_\ell^{-1} u_\ell \rangle, \end{aligned}$$

where $u_a = \varphi_a v_a$. It follows that $\mathcal{Z}(s, P)^{\varphi^{-1}} = \mathcal{Z}(s^\varphi, P) = \mathcal{Z}(t, P)$. Now put $\mu = \varphi^{-1}$ and $\nu = 1$. The reverse implication is proved similarly. \square

2.3. The derivation-densor method. There are many possible ideals, P , one can consider to seed the mechanism in Proposition 2.2. To narrow the candidate pool we impose three requirements: (a) $\mathcal{Z}(t, P)$ has an algebraic structure like $\text{Adj}(t)$ and $\text{Der}(t)$; (b) the choice of P is independent of the given tensor t ; and (c) there is an efficient algorithm to construct $\mathcal{Z}(t, P)$.

For several reasons, the ideal $P = (\mathbf{d})$ with $\mathbf{d} = x_1 + \cdots + x_\ell$ is the perfect candidate. First, $\text{Der}(t) = \mathcal{Z}(t, (\mathbf{d}))$ is the solution space of a system of linear equations, and hence can be constructed efficiently. Secondly, all associative algebras associated to t , such as $\text{Adj}(t)$, embed in $\text{Der}(t)$ [5, Theorem A]. Further, for any ideal P generated by linear homogeneous polynomials, the densor subspace $\langle t \rangle$ embeds in $\mathcal{N}(P, \mathcal{Z}(t, P))$ [14, Theorem A]. *Thus, in a precise sense, $\langle t \rangle$ is the most compressed space one can use with linear methods.*

Since $\text{Der}(t) = \mathcal{Z}(t, (\mathbf{d}))$ and $\langle t \rangle = \mathcal{N}((\mathbf{d}), \mathcal{Z}(t, (\mathbf{d})))$, the correctness of the derivation-densor method follows directly from Proposition 2.2 applied to $P = (\mathbf{d})$:

Theorem 2.3. *Algorithm 1 decides isomorphism of tensors $s, t \in (V_1 \otimes \cdots \otimes V_\ell)^*$.*

3. DECIDING PSEUDO-ISOMORPHISM OF SIMPLE LIE MODULES

We turn now to the key steps in Algorithm 1, namely to the related tasks in Lines 2 and 4. As noted in Section 1.3, it is convenient to translate those tasks into pseudo-isomorphism problems for modules. The purpose of this section is to solve the latter module problems for the classes of algebras we consider in this article. In particular, we will prove Theorem 1.3.

3.1. Three illustrations. In associative and Lie settings, pseudo-isomorphism of modules is a strictly coarser equivalence than isomorphism. An analysis of the associative setting is provided in [11]. To elucidate the differences between module isomorphism and pseudo-isomorphism, and to distinguish the Lie module setting from its associative counterpart, we briefly describe three computational settings. Some challenging obstructions are encountered even in these elementary examples.

3.1.1. Irreducible representations of simple algebras. Consider $L = \mathfrak{sl}_3(K)$ acting in two different ways on K^3 as follows: for $x \in L$ and $v \in K^3$,

$$(3.1) \quad [x, v]_1 = xv \qquad [x, v]_2 = \bar{x}v,$$

where $\bar{x}_{ij} = x_{(3-j+1)(3-i+1)}$ is the transpose along the opposite diagonal. Isomorphism can be decided using the standard theory of weight spaces, as described in [21, Chapter VI]. The highest weight space of the first module, ${}_L V_1$, is $V_\lambda = Ke_1$, where λ has support $h_1 = E_{11} - E_{22}$ in the standard Cartan subalgebra. The highest weight space of the second module, ${}_L V_2$, is the same space but with a different weight, namely $V_{\lambda'} = Ke_1$ where λ' has support $h_2 = E_{22} - E_{33}$. Thus, ${}_L V_1$ and ${}_L V_2$ are non-isomorphic L -modules [21, VI.20.3]. However, $\Phi := I_3$ and $\varphi: x \rightarrow \bar{x}$ is evidently a pseudo-isomorphism ${}_L V_1 \rightarrow {}_L V_2$, which could be termed a “graph-twist” because φ induces an automorphism of the Dynkin diagram of L .

J. Grochow observed in his Ph.D. thesis that, when L is a simple Chevalley Lie algebra, the isomorphism classes of simple L -modules are determined by such graph automorphisms [17]. In that case, one can exhaust the limited number (≤ 6) of Dynkin diagram automorphisms until an appropriate choice for $\varphi: L_1 \rightarrow L_2$ is found to reduce the given pseudo-isomorphism problem to an instance of isomorphism that may be solved by the theory of weight spaces. For this observation to be practicable, one requires efficient algorithms to recognize that a given Lie algebra has a Chevalley basis and to construct one if it does. Fortunately, such algorithms exist [12, 26, 30], so Theorem 1.3 holds when the given Lie algebras are simple.

These sorts of pseudo-isomorphisms between modules of simple Lie algebras have no associative analogue: by the Skolem–Noether theorem, every automorphism of a simple associative algebra is inner.

3.1.2. Completely reducible representations of semisimple algebras. Let $L = \mathfrak{sl}_d(K)^n$, and define two actions on $V^n = K^{dn}$ by

$$(3.2) \quad \begin{aligned} [(x_1, \dots, x_n), (v_1, \dots, v_n)]_1 &= (x_1 v_1, \dots, x_n v_n) \\ [(x_1, \dots, x_n), (v_1, \dots, v_n)]_2 &= (x_{\sigma(1)} v_1, \dots, x_{\sigma(n)} v_n). \end{aligned}$$

where σ is a permutation of $[n] := \{1, \dots, n\}$. For large values of n it would be prohibitively expensive to list all permutations as we did with the Dynkin diagram automorphisms. However, more thoughtful strategies also have their limitations: Grochow proved that pseudo-isomorphism in this setting is at least as hard as the Graph Isomorphism problem [17].

There is an analogous situation for associative algebras—where it is equally futile to list all possible permutations—but an efficient solution exists in this case [11, Theorem 1.3].

3.1.3. Irreducible representations of semisimple algebras. For $i \in [m]$, let E_i be a field extension of K . Consider $L = \mathfrak{sl}_{d_1}(E_1) \oplus \cdots \oplus \mathfrak{sl}_{d_m}(E_m)$ acting on $V = E_1^{d_1} \otimes \cdots \otimes E_m^{d_m}$ by

$$(3.3) \quad [(x_1, \dots, x_\ell), v_1 \otimes \cdots \otimes v_m] = x_1 v_1 \otimes \cdots \otimes x_\ell v_m,$$

where $x_i \in \mathfrak{sl}_{d_i}(K_i)$, $v_i \in E_i^{d_i}$. Again, permutations of coordinates threatens to encode a hard problem as an instance of a pseudo-isomorphism problem of this type. However, unlike Section 3.1.2, we have a tensor product rather than a direct sum, so minimal ideals of L do not annihilate subspaces of the module.

This situation is again particular to the nonassociative setting.

3.2. Tensor decompositions of simple Lie modules. The situation described in Section 3.1.3 illustrates a general phenomenon. If $L = M \oplus N$ is a Lie algebra decomposition into ideals, and V is an L -module, then a consequence of the Jacobi identity is that $L = M \oplus N$ acts on V in the following sense:

$$(3.4) \quad (\forall m \in M)(\forall n \in N)(\forall v \in V) \quad m(nv) = n(mv).$$

This property enables us to characterize, constructively as iterated tensor products, the Lie modules arising in Theorem 1.3.

For an ideal M of $L \leq \mathfrak{gl}(V)$, let $K\langle M \rangle \leq \text{End}(V)$ denote its associative envelope, namely the K -span of the semigroup generated by $M \subseteq \text{End}(V)$. For $S \subset V$, put $MS := K\langle M \rangle S$, the smallest M -submodule of V containing S . The following elementary result provides the engine for our decomposition algorithm.

Lemma 3.5. *Let $L = M \oplus N$ be a nontrivial decomposition with M a minimal ideal, and let V be a simple L -module. If S is a proper, simple M -submodule of V , then $V = S \oplus NS$ is an M -module decomposition, and S embeds in NS .*

Proof. For an M -submodule, S , of V , we have $M(NS) = N(MS) \leq NS$ by (3.4), so NS is an M -submodule. As V is a simple L -module, $V = LS = MS + NS \leq S + NS$. Since S is a proper, simple M -module, $S \cap NS = 0$ as required. Since S is not an L -submodule of V , it follows that $NS \neq 0$, so there exists $n \in N$ with $nS \neq 0$. By (3.4) again, $s \mapsto ns$ is an M -module embedding $S \rightarrow NS$. \square

To ensure linear algebra is done over fields rather than division rings, we introduce an additional condition on minimal ideals M of a Lie algebra L . An ideal M of L has *central type* if $K\langle M \rangle$ is isomorphic to $\mathbb{M}_f(\Delta)$ for some integer f and field Δ . We say L has *central type* if all minimal ideals of L have central type.

Lemma 3.6. *If M is a simple Lie algebra of central type and S a simple faithful M -module, then $\text{End}_M(S)$ is a field.*

Proof. By Schur's lemma, $\Delta = \text{End}_M(S)$ is a division ring, and by Wedderburn–Artin, $M \cong \text{End}_\Delta(S)$. Since M is of central type, Δ is a field. \square

Our decomposition algorithm also makes use of idempotents of a matrix algebra over K . An element $e \in A \leq \text{End}(V)$ is a *idempotent* if $e^2 = e$. Two idempotents $e, f \in A$ are *orthogonal* if $ef = 0 = fe$. Finally, an idempotent $e \in A$ is *primitive* if e is not the sum of two nonzero orthogonal idempotents.

Theorem 3.7. *There is a polynomial-time algorithm that, given a decomposition $L = M \oplus N$, with M minimal and of central type, and a simple L -module V , returns*

an M -submodule $S \leq V$, an N -submodule $T \leq V$, and an L -module isomorphism $V \rightarrow S \otimes_{\text{End}_M(S)} T$.

Proof. There are two stages to the algorithm. The first constructs a decomposition $V \cong S \otimes_{\Delta} \Delta^r$ for a simple M -submodule S , where $\Delta = \text{End}_M(S)$. The second constructs an N -submodule $T \leq V$ and an isomorphism $T \rightarrow \Delta^r$.

We first verify, in polynomial time, that M is minimal by checking that $K\langle M \rangle$ is simple using [29]. Then we apply polynomial-time algorithms from [19, 29] to test whether V is a simple M -module. If V is simple, set $S = V$. Otherwise the algorithms of [19, 29] produce a proper, simple M -submodule $S \leq V$.

Initialize $\mathcal{V} := \{S\}$. While $\sum_{U \in \mathcal{V}} U \neq V$, find a generator $n \in N$ such that $nS \cap \sum_{U \in \mathcal{V}} U = 0$, and put $\mathcal{V} := \mathcal{V} \cup \{nS\}$. By the proof of Lemma 3.5, S and nS are isomorphic M -submodules of V , so we obtain an isomorphism $\alpha: V \rightarrow S^{\oplus r}$ of M -modules. By solving a system of linear equations, we construct generators for the field $\Delta = \text{End}_M(S)$, cf. Lemma 3.6. We use α to write $V \cong S \otimes_{\Delta} \Delta^r$. This completes the first stage of the algorithm.

We begin the second stage by computing generators for $A := K\langle M \rangle \subset \text{End}_K(V)$. We next use the recognition algorithm of [23, Theorem 1] to construct a primitive idempotent $e \in A$, and put $T := eV$. Since e is a primitive idempotent of $A \cong \mathbb{M}_f(\Delta)$, there is a natural ring isomorphism $eAe \rightarrow \Delta$, and a Δ -vector space isomorphism $\Delta \rightarrow eS$. Hence,

$$\begin{aligned} S \otimes_{\Delta} T &\cong S \otimes_{\Delta} (eS \otimes_{\Delta} \Delta^r) \\ &\cong S \otimes_{\Delta} (\Delta \otimes_{\Delta} \Delta^r) \\ &\cong S \otimes_{\Delta} \Delta^r \cong V. \end{aligned} \quad \square$$

The relevant mappings in the above isomorphisms are a by-product of the computation as well.

Corollary 3.8. *Let $L = M_1 \oplus \cdots \oplus M_r$ be a decomposition into nontrivial minimal ideals of central type, and let V be a simple L -module. Then $V \cong S_1 \otimes \cdots \otimes S_r$, where $S_i \leq V$ is a simple M_i -submodule for $i \in [r]$. Furthermore, if L is of central type and the decomposition of L into ideals is given, then the tensor decomposition of V can be constructed in polynomial time.*

One final ingredient is needed for the proof of Theorem 1.3. For finite fields one can, in polynomial time, test whether a given (unital) associative algebra is cyclic (a quotient of a polynomial ring), and decide pseudo-isomorphism of modules for such algebras [11]. Those algorithms have since been generalized to number fields [23, pp. 211–212]; the next result follows *mutatis mutandis* from [11, Theorem 1.3].

Theorem 3.9. *Fix a field K that is finite or finite over \mathbb{Q} , and a finite-dimensional vector space V . There is a polynomial-time algorithm to decide if an associative algebra $A \leq \text{End}_K(V)$ is cyclic and another to settle pseudo-isomorphism for modules of such algebras.*

3.3. Proof of Theorem 1.3. Let L_1 and L_2 be the given Lie algebra of Chevalley type represented faithfully on simple modules V_1 and V_2 , respectively. For $i = 1, 2$, L_i is reductive (it has central nil radical), so it has central type and decomposes as $L_i = M_{i0} \oplus M_{i1} \oplus \cdots \oplus M_{ir_i}$ into nontrivial minimal ideals, with M_{i0} abelian. Moreover, such a decomposition can be found using [23, Theorem 1] and the more general finite field case discussed in [23, pp. 211–212]. We may assume $r_1 = r_2 = r$

since, otherwise, $L_1 \not\cong L_2$. By re-indexing we may further assume, for each $k \in [r]$, that $M_{1k} \cong M_{2k}$ as Lie algebras by computing Chevalley bases—possibly over extension fields—and comparing root data. For the abelian ideals M_{10} and M_{20} , we simply compare dimensions.

We first handle the abelian ideals. By Schur’s Lemma, $K\langle M_{10} \rangle$ and $K\langle M_{20} \rangle$ are both cyclic algebras. Using Theorem 3.9, we construct $\psi_0: K\langle M_{10} \rangle \rightarrow K\langle M_{20} \rangle$ and $\Psi_0: V_1 \rightarrow V_2$ such that $(\Psi_0, \psi_0 \oplus \text{id}_1 \oplus \cdots \oplus \text{id}_r)$ is a pseudo-isomorphism $K\langle M_{10} \rangle V_1 \rightarrow K\langle M_{20} \rangle V_2$.

Next, for each $i \in [2]$, apply Corollary 3.8 to construct a tensor decomposition $V_i = S_{i1} \otimes \cdots \otimes S_{ir}$, where S_{ij} a simple M_{ij} -module for $j \in [r]$. For each $j \in [r]$, use Grochow’s algorithm [17] (discussed in Section 3.1.1) to construct a pseudo-isomorphism (Ψ_j, ψ_j) from ${}_{M_{1j}}S_{1j} \rightarrow {}_{M_{2j}}S_{2j}$. If the latter fails for some j , then there is no pseudo-isomorphism ${}_{L_1}V_1 \rightarrow {}_{L_2}V_2$, so we report that and exit. Otherwise, $((\Psi_1 \otimes \cdots \otimes \Psi_r) \cdot \Psi_0, \psi_0 \oplus \psi_1 \cdots \oplus \psi_r)$ is the desired pseudo-isomorphism ${}_{L_1}V_1 \rightarrow {}_{L_2}V_2$. \square

4. FAMILIES OF DENSORS WITH PRESCRIBED DIMENSIONS

In this section, we construct an infinite family of tensors with small tensor spaces. In particular, there is a sub-family whose tensor spaces are 1-dimensional. These tensors come from the classical representation theory of \mathfrak{sl}_n -modules, and we expect that similar ideas can be used to build more such families.

Throughout this section, K will denote a field that is either finite or finite over \mathbb{Q} . Let n be a positive integer such that if $\text{char}(K) = p > 0$ then $p \nmid (n+1)$. Let $L = \mathfrak{sl}_{n+1}(K)$, the simple Lie algebra of type A_n , and let M be a finite-dimensional simple L -module. The Lie module operation is a K -bilinear map, $\langle t | : L \times M \rightarrow M$, and $\delta = (\delta_2, \delta_1, \delta_0)$ is a *derivation* of t if, for all $x \in L$ and $v \in M$,

$$(4.1) \quad \delta_0 \langle t | x, v \rangle = \langle t | \delta_1 x, v \rangle + \langle t | x, \delta_2 v \rangle.$$

Equivalently, we could construct from t a trilinear form, giving rise to a natural bijection between the derivations in (4.1) and the derivations defined in Section 1.

Lemma 4.2. *Der(t) contains a simple subalgebra isomorphic to $\mathfrak{sl}_{n+1}(K)$.*

Proof. Since M is an L -module, it follows that for all $v \in M$ and $x, y \in L$,

$$\langle t | xy, v \rangle = (xy)v = x(yv) - y(xv) = x\langle t | y, v \rangle - \langle t | y, xv \rangle.$$

Therefore, L embeds into $\text{Der}(t)$, and the lemma follows since $L \cong \mathfrak{sl}_{n+1}(K)$. \square

The simple L -module M contains a unique vector of highest weight λ . We write $M = V(\lambda)$ if M is an L -module with highest weight λ , where λ is a partition with n parts, possibly equal to 0. Write $\lambda = (\lambda_1, \dots, \lambda_n) \vdash m$ if $\sum_i \lambda_i = m$. We need to determine the number of irreducible submodules of $V(\lambda) \otimes V(\mu)$ isomorphic to $V(\nu)$, which are the Littlewood–Richardson numbers for type A , denoted by $c_{\lambda, \mu}^{\nu}$. These numbers can be computed by algorithms on Young tableaux, similar to the well-known \mathfrak{gl}_n case. We follow closely the notation used in [20].

We denote by Y a Young diagram of type $\lambda \vdash m$. Let $\mathcal{B}(Y)$ be the set of *semi-standard Young tableaux* obtained by filling in the boxes of the diagram Y with integers $[n+1]$ such that each row is weakly increasing and each column is strictly increasing. A tableau is *standard* if the integers 1 through m appear once.

For a Young diagram Y of type $\lambda = (\lambda_1, \dots, \lambda_n)$, define a new Young diagram

$$(4.3) \quad Y[j] = \begin{cases} (\lambda_1, \dots, \lambda_j + 1, \dots, \lambda_n) & j \leq n, \\ (\lambda_1 - 1, \dots, \lambda_n - 1) & j = n + 1. \end{cases}$$

For $m \geq 2$, we define $Y[b_1, \dots, b_{m-1}, b_m]$ recursively so that

$$Y[b_1, \dots, b_{m-1}, b_m] = Y[b_1, \dots, b_{m-1}][b_m],$$

provided $Y[b_1, \dots, b_i]$ is a Young diagram for all $i \in [m-1]$. The next theorem states how this operation can be used to determine $c_{\lambda, \mu}^{\lambda'}$.

For a Young diagram of type λ with n parts, (a basis of) the weight space decomposition of $V(\lambda)$ corresponds to the set $\mathcal{B}(Y)$. We abuse notation and identify the two, working with tableaux instead. So we write $\mathcal{B}(Y) \oplus \mathcal{B}(Y')$, the disjoint union of tableaux, for the module $V(\lambda) \oplus V(\lambda')$.

Theorem 4.4 ([20, Theorem 8.6.6]). *Let λ and μ be partitions with n parts, and let Y and Y' be the corresponding Young diagrams. Then there exists an isomorphism of \mathfrak{sl}_{n+1} -modules*

$$\mathcal{B}(Y) \otimes \mathcal{B}(Y') \cong \bigoplus_{b_1 \otimes \dots \otimes b_m \in \mathcal{B}(Y')} \mathcal{B}(Y[b_1, \dots, b_m]).$$

Note, if $Y[b_1, \dots, b_m]$ is not a Young diagram, then $\mathcal{B}(Y[b_1, \dots, b_m]) = 0$.

Proposition 4.5. *With $n \geq 1$, set $\mu = (2, 1, \dots, 1) \vdash n + 1$. If $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition, then $c_{\lambda, \mu}^{\lambda} = |\{\lambda_i \mid 1 \leq i \leq n, \lambda_i > 0\}|$.*

Proof. Write $\lambda = (n_1, \dots, n_1, n_2, \dots, n_2, \dots, n_k, \dots, n_k)$ for some $k \leq n$, where $n_i > n_{i+1}$ for $i \in [k-1]$. Let Y and Y' be the Young diagrams corresponding to λ and μ respectively. We count the number of summands equal to $\mathcal{B}(Y)$ in $\mathcal{B}(Y) \otimes \mathcal{B}(Y')$. From Theorem 4.4 these correspond to tableaux $T := b_1 \otimes \dots \otimes b_{n+1} \in \mathcal{B}(Y')$ such that $Y[b_1, \dots, b_{n+1}] = Y$. The latter condition implies that T is a standard Young tableau of type $\mu = (2, 1, \dots, 1)$, so $b_2 = 1 \neq b_1$.

If $n_k = 0$, since $b_1 \neq n+1$ there are $k-1$ choices for b_1 such that $Y[b_1]$ is a Young tableau. If $n_k > 0$, there are k choices for b_1 . Since $1 = b_2 < b_3 < \dots < b_{n+1} \leq n+1$, the remaining b_i in both cases are uniquely determined. Thus, $c_{\lambda, \mu}^{\lambda} \in \{k-1, k\}$ depending only on whether $n_k = 0$ or $n_k > 0$; the result follows. \square

Using these results we can now build families of tensors with 1-dimensional tensors. If m is another positive integer, let $d(m, n)$ be the number of divisors of m no larger than n .

Theorem 4.6. *For any K there are infinitely many positive integers n such that, for all positive integers m , there are at least $d(m, n)$ pairwise non-isomorphic K -tensors with 1-dimensional tensor space.*

Proof. Set $L = \mathfrak{sl}_{n+1}(K)$. By Lemma 4.2, $\text{Der}(t)$ contains a simple subalgebra $D \leq \text{Der}(t)$ isomorphic to L . Setting $\mathbf{d} = x_2 + x_1 - x_0$, we consider only $\mathcal{N}(\mathbf{d}, D)$ in place of $\mathcal{N}(t, D)$. Note that $\mathcal{N}(t, D) \leq \mathcal{N}(\mathbf{d}, D)$. We will show that $\dim \mathcal{N}(\mathbf{d}, D) = 1$, so that $\mathcal{N}(t, D) = \mathcal{N}(\mathbf{d}, D)$ as $0 \neq t \in \mathcal{N}(t, D)$.

Since M and L are irreducible L -modules, they are irreducible D -modules. Every tensor contained in $\mathcal{N}(t, D)$ determines a $\text{Der}(t)$ -module homomorphism $M \rightarrow M \otimes L$, which must also be a D -module homomorphism. Each irreducible L -module has a unique vector of highest weight, so there exist partitions λ and μ such that

$M \cong V(\lambda)$ and $L \cong V(\mu)$ as D -modules. Since L is the adjoint module, $\mu = (2, 1, \dots, 1) \vdash n + 1$. By irreducibility, the number of K -linearly independent D -module homomorphisms $M \rightarrow M \otimes L$ is equal to the generalized Littlewood–Richardson number for type A , namely $c_{\lambda, \mu}^{\lambda}$. For $m \geq 1$ and for all positive integers ℓ such that $\ell \mid m$ and $\ell \leq n$, let $\lambda \vdash m$ with parts of size ℓ and 0. From Proposition 4.5, $c_{\lambda, \mu}^{\lambda} = 1$. There are at least $d(m, n)$ such partitions λ , which proves the theorem. \square

5. PROOF OF THEOREM 1.4

Let $t \in (V_1 \otimes \dots \otimes V_\ell)^*$ be nondegenerate, and assume $L := \text{Der}(t)$ is reductive.

First we show that if some V_a is non-simple as an L -module then $\dim \mathfrak{t} > 1$. Suppose, for some $a \in [\ell]$, that U_a is a proper nontrivial L -submodule of V_a . Let $e: V_a \rightarrow V_a$ be an idempotent with kernel U_a , and set $\langle s|v \rangle = \langle t|v_1, \dots, ev_a, \dots, v_\ell \rangle$. Since L is reductive, the image of e is an L -module complement to U_a . Thus, for each $\delta \in L$, $e\delta_a = \delta_a e$. It follows that $L \subseteq \text{Der}(s)$, and hence that $s \in \mathfrak{t}$. Because U_a is nontrivial and proper, s is nonzero and degenerate. Since t is nondegenerate, s and t are linearly independent vectors so $\dim \mathfrak{t} > 1$.

Next, let $t_1, t_2 \in (V_1 \otimes \dots \otimes V_\ell)^*$ be two nondegenerate tensors having 1-dimensional densor spaces. We apply Algorithm 1 to test for isomorphism.

First, the derivation algebras $L_i := \text{Der}(t_i)$ (Line 1) are constructed in polynomial time by solving a linear system. By assumption, each L_i is reductive which allows us to decompose $L_i = M_{i1} \oplus \dots \oplus M_{ir_i}$ into nontrivial minimal ideals (see [23, Theorem 1] and the more general finite field case discussed in [23, pp. 211–212]). If $r_1 \neq r_2$, then $\text{Der}(t_1)$ is not conjugate to $\text{Der}(t_2)$.

As $\dim \mathfrak{t}_i = 1$, for each $a \in [\ell]$, V_a is a simple L_i -module ($i = 1, 2$). So we may apply Theorem 1.3 to construct $\varphi_a \in \text{GL}(V_a)$ such that $(L_1|_{V_a})^{\varphi_a} = L_2|_{V_a}$. The action of $L_i = M_{i1} \oplus \dots \oplus M_{ir_i}$ on $V_1 \otimes \dots \otimes V_\ell$ satisfies the property in (3.4), so setting $\varphi := \varphi_1 \otimes \dots \otimes \varphi_\ell$, gives $L_1^\varphi = L_2$ in $\text{End}((V_1 \otimes \dots \otimes V_\ell)^*)$. This completes Line 2 of Algorithm 1.

Since $\dim \mathfrak{t}_i = 1$, we do not need to induce images of normalizers. Therefore, we proceed to Line 4, where the task is merely to decide if $t_1^\varphi = \lambda t_2$ for some scalar λ . This is settled by solving a tiny linear equation, so the result follows. \square

Remark 5.1. Theorem 1.4 decides isomorphism within the family of tensors in Theorem 4.6 in polynomial time, but we are aware of no other sub-exponential isomorphism tests for this family. For instance, if t is a tensor in this family, a consequence of the construction is that $\text{Adj}(t) \cong K$. Thus, the adjoint-tensor method is no better than brute force for this family of tensors.

6. FURTHER RESULTS

There are a number of similar results attainable by modest adaptation of our methods. We are careful to avoid constructing $N(\text{Der}(t))$ for general fields K , since K^\times may not have a finite generating set. When K is finite, however, we can give generators for K^\times and, consequently, also for $N(\text{Der}(t))$.

Theorem 6.1. *Let K be a finite field with $K = 6K$, and let $t \in (K^{d_1} \otimes \dots \otimes K^{d_\ell})^*$ satisfy the hypotheses of Theorem 1.4. In polynomial time one can construct generators for the group $\text{Aut}(t)$.*

When $\dim \langle t \rangle > 1$ it is still possible that $\text{Der}(t)$ is reductive and irreducible on each V_a . In that case we are left to search the orbit of $N(\text{Der}(t))$ acting on $\langle t \rangle$.

When $\text{Der}(t)$ is represented *reducibly* on the V_a , one is confronted with familiar difficulties when matching simple factors. Indeed, Grochow has shown that a general solution to the conjugacy problem for semisimple Lie algebras over any field requires solving Graph Isomorphism [17]. However, this obstruction is not so formidable when the number of simple $\text{Der}(t)$ -modules is bounded.

The situation when $\text{Der}(t)$ has a noncentral nil radical is worse. Indeed, the existence of radicals is a problem even for associative algebras [10]. Although the presence of a flag suggests that an inductive process may succeed, all actions must also normalize the radical. This extra condition is itself a tensor isomorphism problem, but now involving tensors that arise as the product of the nil radical. It is not known if this case is as hard as the general case of tensor isomorphism, but certainly no efficient solution is known.

Although the derivation algebras of tensors over fields of positive characteristic are restricted Lie algebras, they can have (nonabelian) simple factors that are not of Chevalley type. For example, let $A = \mathbb{F}_p[x]/(x^p)$, for a prime p , and define $\langle t \rangle: A^2 \times A^2 \rightarrow A$ via

$$\langle t \rangle(a, b), (x, y) = ay - bx.$$

This tensor can also be interpreted as the commutator of the Heisenberg group $H(A)$. The derivation algebra of t is isomorphic to $\text{Der}(A) \oplus \mathfrak{sl}_2(A) \oplus A^2$, where $\text{Der}(A)$ is the simple p -dimensional Jacobson–Witt Lie algebra of derivations of A . Over \mathbb{F}_p , the tensor appears to have a 1-dimensional tensor subspace for some small primes. By Corollary 3.8, we can extend Theorem 1.4 to a broader class, \mathcal{C} , of restricted Lie algebras, provided we have a polynomial-time algorithm to decide pseudo-isomorphism of simple modules over simple Lie algebras in \mathcal{C} .

We have implemented prototypes of our algorithms in the MAGMA system [2]. They are publicly available within software packages for effective computation with tensors [7].

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